

# SHARP LORENTZ SPACE ESTIMATES FOR ROUGH OPERATORS

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**ABSTRACT.** We demonstrate the  $(H^1, L^{1,2})$  or  $(L^p, L^{p,2})$  mapping properties of several rough operators. In all cases these estimates are sharp in the sense that the Lorentz exponent 2 cannot be replaced by any lower number.

## 1. Introduction

In this paper we consider the endpoint behaviour on Hardy spaces of two classes of operators, namely singular integral operators with rough homogeneous kernels [4] and singular integral operators with convolution kernels supported on curves in the plane ([20], [27]). These operators fall outside the Calderón-Zygmund theory; however weak type  $(L^1, L^{1,\infty})$  or  $(H^1, L^{1,\infty})$  inequalities have been established in the previous literature ([7], [9], [16] [18], [25], [29]). We shall show that the target space  $L^{1,\infty}$  can be improved to the Lorentz space  $L^{1,2}$ , possibly at the cost of moving to a stronger type of Hardy space (*e.g.* product  $H^1$ ). Examples of Christ [8], [17] show that these types of results are optimal in the sense that one cannot replace  $L^{1,2}$  by  $L^{1,q}$  for any  $q < 2$ .

The space  $L^{1,2}$  arises naturally as the interpolation space halfway between  $L^{1,\infty}$  and  $L^1$ . As a gross caricature of how this space arises, suppose that we have a collection of functions  $f_i$  which are uniformly bounded in  $L^1$ , and whose maximal function  $\sup_i |f_i|$  is in weak  $L^1$ , and we wish to estimate the quantity

$$\left\| \sum_i \gamma_i f_i \right\|_{L^{1,2}}$$

for some  $l^2$  co-efficients  $\gamma_i$ . If the  $f_i$  are sufficiently orthogonal, we may hope to control this quantity by the square function

$$(1.1) \quad \left\| \left( \sum_i |\gamma_i f_i|^2 \right)^{1/2} \right\|_{L^{1,2}}.$$

However from our hypotheses we see that

$$\left\| \left( \sum_i |\gamma_i f_i|^q \right)^{1/q} \right\|_{L^{1,q}} \lesssim \left( \sum_i |\gamma_i|^q \right)^{1/q}$$

for  $q = 1$  and  $q = \infty$ , and thus by interpolation for all  $1 \leq q \leq \infty$  (*cf.* Lemma 2.2. below). Thus we expect to control (1.1) by the  $\ell^2$  norm of  $\{\gamma_i\}$ .

Our arguments will be based on more complicated versions of the above informal strategy. Generally, the  $L^1$  estimates will be quite trivial, whereas the  $L^{1,\infty}$  estimates will be variants of existing weak-type (1,1) estimates for rough operators in the literature (*e.g.* [7], [25]). We shall demonstrate this technique for

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two classes of operators. Firstly we show that the Hilbert transform on plane curves  $(t, t^m)$  maps product  $H^1$  into  $L^{1,2}$  or a related Hardy-Lorentz space; we also prove sharp  $L^p \rightarrow L^{p,2}$  estimates for a related analytic family of hypersingular operators. Then we discuss homogeneous singular integrals with rough kernels in  $\mathbb{R}^d$ , satisfying an  $L \log^2 L$  condition on the sphere, and show that these map the standard Hardy space  $H^1$  to  $L^{1,2}$ .

We remark that a simple version of the above technique has been used by one of the authors in [23] to prove an endpoint version of the Hörmander multiplier theorem. Namely (stating only the one-dimensional version) if  $\phi$  is a nonzero even smooth bump function then the condition  $\sup_{t>0} \|\phi m(t\cdot)\|_{B_{1/2,1}^2}$  implies that the convolution operator with Fourier multiplier  $m$  maps  $H^1$  to  $L^{1,2}$  (and an example by Baernstein and Sawyer [1] shows that  $L^{1,2}$  cannot be replaced by  $L^{1,q}$  for  $q < 2$ ). The second author and Jim Wright [30] have recently improved this result by replacing the Besov space  $B_{1/2,1}^2$  by the larger space  $R_{1/2,2}^2$  defined in [24] improving on the known  $(H^1, L^{1,\infty})$  result which is implicit in the latter paper.

The paper is structured as follows. After formulating our results in the current section we review some material about Hardy-Lorentz spaces and interpolation, in §2. In §3 we prove an abstract variant of a stopping time argument due to M. Christ which may be helpful elsewhere. §4 contains the main square-function estimate needed to prove our theorems on integrals along curves; in §5 we conclude the proof of these results. Rough homogeneous kernels are considered in §6 and §7.

### Rough homogeneous convolution kernels.

Let  $K$  be a convolution kernel on the Euclidean space  $\mathbb{R}^d$  and assume that  $K$  is homogeneous of degree  $-d$  and that the restriction  $\Omega$  to the unit sphere is integrable and has mean zero,  $\int_{S^{d-1}} \Omega(\theta) d\sigma(\theta) = 0$ . We may define the operator  $T_\Omega$  of convolution with  $K$  on test functions at least by the usual method of principal values:

$$(1.2) \quad T_\Omega f(x) = p.v. \int \frac{\Omega(y/|y|)}{|y|^d} f(x-y) dy.$$

We consider the mapping properties of  $T_\Omega$ , especially near the endpoint  $L^1$ . If  $\Omega$  is somewhat regular (for example, if it is Hölder continuous or satisfies an appropriate  $L^1$  Dini condition) then the standard Calderón-Zygmund theory shows that  $T$  is bounded on all  $L^p$  spaces,  $1 < p < \infty$ , is of weak type  $(1,1)$ , and maps the Hardy space  $H^1$  to  $L^1$ . If no regularity is assumed, but  $K$  is  $L \log L$  on the sphere, then it was shown by Calderón-Zygmund [4] that  $T_\Omega$  is bounded on  $L^p$ ; in fact (see [25]) it is of weak type  $(1,1)$ . The behaviour at  $H^1$  is more subtle, however, as an example of M. Christ shows (see also [17]). For the sake of illustration let us consider the case  $d = 2$ . Let  $a$  be a smooth  $H^1$  atom on the unit ball, which is smooth and radial, and let  $\Omega_N$  be the lacunary function defined on the unit circle by

$$\Omega_N(\cos \alpha, \sin \alpha) \equiv G_N(\alpha) = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{2\pi i C^j \alpha},$$

where  $C, N$  are large integers. Roughly speaking, the function  $K * a(x)$  has magnitude  $\sim N^{-1/2} |x|^{-d}$  whenever  $|x| \sim C^j$  for some  $j = 1, \dots, N$ . This shows that the  $L^1$  norm (and indeed the  $L^{1,q}$  quasi-norm for any  $q < 2$ ) of  $K * a$  grows with  $N$ , even though  $\Omega$  is in every  $L^p$  class,  $p < \infty$ , uniformly in  $N$ . Thus, the best result one can reasonably hope for is that  $T$  maps  $H^1$  to the Lorentz space  $L^{1,2}$ , or the Hardy-Lorentz space  $H^{1,2}$ , the quasi-norm norm in the latter is the  $L^{1,2}$  quasinorm of a suitable square-function or maximal operator used in the definition of  $H^1$  (see §2 below).

The previous counterexample can be modified to include the case  $\Omega \in L^\infty$ . Take  $G_N$  as above,  $\varepsilon > 0$  and let  $E_{\varepsilon,N} = \{\alpha : |G_N(\alpha)| > N^\varepsilon\}$ . Define  $G_{\varepsilon,N}(\alpha) = (G_N(\alpha)(1 - \chi_{E_{\varepsilon,N}}(\alpha)))$  and

$$\Omega_{\varepsilon,N}(\cos \alpha, \sin \alpha) = G_{\varepsilon,N}(\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} G_{\varepsilon,N}(s) ds.$$

Since  $G_N$  is in  $BMO$  with norm independent of  $N$  we have by the John-Nirenberg inequality that  $|E_{\varepsilon,N}| = O(e^{-cN^\varepsilon})$ , for some  $c > 0$ . From this one checks that the  $L^1$  norm of  $T_{\Omega_N - \Omega_{N,\varepsilon}} a$  over the annulus  $|x| \sim C^j$  is  $O(N^{1/2}e^{-cN^\varepsilon} + 2^{-j})$ , hence negligible. Since on the other hand  $\|\Omega_{N,\varepsilon}\|_\infty \lesssim N^\varepsilon$  this disproves a uniform  $H^1 \rightarrow L^{1,q}$  estimate for  $q < 2/(1+2\varepsilon)$ .

**Theorem 1.1.** *Let  $\Omega \in L \log^2 L(S^{d-1})$  and assume that  $\int_{S^{d-1}} \Omega d\sigma(\theta) = 0$ . Then the operator  $T_\Omega$  maps  $H^1$  to  $H^{1,2}$  and also to  $L^{1,2}$ .*

*Remark 1.2.* In fact we shall see that the  $L \log^2 L$  condition can be strengthened to an  $L \log L$  condition for a Littlewood-Paley square function (see Theorem 6.1 below)

Analogously we may also consider a maximal variant of  $T$ ; here no cancellation is imposed. Let  $\chi \in C_0^\infty(\mathbb{R}^d)$  and

$$(1.3) \quad \mathcal{M}_\Omega f(x) = \sup_{h>0} \left| \int \frac{1}{h^d} \chi\left(\frac{y}{h}\right) \Omega\left(\frac{y}{|y|}\right) f(x-y) dy \right|.$$

**Theorem 1.3.** *Let  $\Omega \in L \log^2 L(S^{d-1})$ . Then  $\mathcal{M}_\Omega$  maps  $H^1$  to  $L^{1,2}$ .*

Again, a modification of the above example shows that  $\mathcal{M}_\Omega$  may fail to map  $H^1$  into  $L^{1,q}$  for  $q < 2$ .

### Integrals along curves in the plane.

In this subsection we shall always be working in the plane  $\mathbb{R}^2$ . Let  $m > 1$  be a real number; all constants may implicitly depend on  $m$ .

Define the Hilbert transform  $Hf$  and the maximal function  $Mf$  along the curve  $(t, |t|^m)$  by

$$(1.4) \quad Hf(x) = p.v. \int f(x_1 - t, x_2 - |t|^m) \frac{dt}{t}$$

and

$$(1.5) \quad Mf(x) = \sup_{h>0} \left| \int f(x_1 - t, x_2 - |t|^m) \frac{1}{h} \eta\left(\frac{t}{h}\right) dt \right|;$$

here  $\eta$  is a smooth function with compact support. These operators are invariant with respect to the scaling

$$(1.6) \quad (x_1, x_2) \mapsto (tx_1, t^m x_2), \quad t > 0.$$

We shall work with the product type Hardy space on  $\mathbb{R}^2$ , considered by Chang and Fefferman [6] among others; we denote this space by  $H_{prod}^1$ . Moreover we denote by  $H_{prod}^{1,2}$  the product-type Hardy-Lorentz space (see §2).

**Theorem 1.4.**  *$\mathcal{M}$  maps  $H_{prod}^1$  to  $L^{1,2}$ , and  $H$  maps  $H_{prod}^1$  to  $H_{prod}^{1,2}$  and to  $L^{1,2}$ .*

This should be compared with the results of Christ [7] who showed that  $M$  and  $H$  map the one-parameter Hardy space  $H_{parabolic}^1$  (defined with respect to the dilations (1.6)) to  $L^{1,\infty}$ , see also Grafakos [16]. In fact, Christ [7] observes that  $H_{parabolic}^1$  is not mapped to  $L^{1,q}$  for  $q < \infty$ .

Now let  $\gamma = (\gamma_1, \gamma_2)$  be a complex multi-index with  $\operatorname{Re}(\gamma_1), \operatorname{Re}(\gamma_2) \geq 0$ , and define the (pseudo)-differentiation operator  $\mathcal{D}^\gamma$  by

$$\widehat{\mathcal{D}^\gamma f} = |\xi^\gamma| \hat{f} = |\xi_1|^{\gamma_1} |\xi_2|^{\gamma_2} \hat{f}.$$

Consider the family of hypersingular operators  $H_\gamma$  defined by

$$(1.7) \quad H_\gamma f(x_1, x_2) = p.v. \int_{-\infty}^{\infty} \mathcal{D}^\gamma f(x_1 - t, x_2 - |t|^m) |t|^{\gamma_1 + \gamma_2 m} \frac{dt}{t}.$$

The space  $L^p$  ( $1 < p < 2$ ) is not mapped to  $L^{p,q}$  if  $q < 2$  (see [8]); moreover this shows that  $H$  does not map  $H_{prod}^1$  to  $L^{1,q}$  or any Hardy-Lorentz space  $H^{1,q}$  for any  $q < 2$ . An angular Littlewood-Paley theory plays a role in this counterexample. Grafakos [16] proved using the methods in [7] that for  $m = 2$ ,  $\gamma_1 = 0$  and  $\operatorname{Re}(\gamma_2) = 1 - 1/p$  the space  $L^p$  is mapped to  $L^{p,p'}$  if  $1 < p \leq 2$ . His method surely extends to the general case considered here.

An improved optimal result is

**Theorem 1.5.** *Suppose that  $\operatorname{Re}(\gamma_1) \geq 0$ ,  $\operatorname{Re}(\gamma_2) \geq 0$  and  $\operatorname{Re}(\gamma_1 + \gamma_2) = 1 - 1/p$ .*

- *If  $1 < p \leq 2$  then  $H_\gamma$  is bounded from  $L^p$  to  $L^{p,2}$ .*
- *If  $p = 1$  then  $H_\gamma$  is bounded from  $H_{prod}^1$  to  $L^{1,2}$ .*

*In both cases the bounds grow at most polynomially in  $|\gamma|$ .*

The following estimate for a localized averaging operator will follow from our proof. Let  $\eta \in C_0^\infty(\mathbb{R})$  and define

$$(1.8) \quad Af(x_1, x_2) = \int \eta(t) f(x_1 - t, x_2 - |t|^m) dt.$$

**Corollary 1.6.** *Suppose  $m \geq 2$ . Then  $A$  maps  $L^{m,2}$  boundedly to the Sobolev space  $L_{1/m}^m$ .*

*Remarks 1.7.*

(i) Suppose that  $t \mapsto g(t)$  is a smooth curve passing through the origin and suppose that its curvature vanishes to at most order  $m - 2$  at the origin. Then the statement of Corollary 1.8 remains true if  $(t, |t|^m)$  is replaced by a  $g(t)$  provided that  $\eta$  is supported in a sufficiently small neighborhood of the origin.

(ii) In the statements of Theorems 1.4 and 1.5 the curve  $(t, |t|^m)$  can be replaced by  $(t, |t|^m \operatorname{sign}(t))$ .

(iii) A variant of this family  $H_\gamma$  was previously considered by Stein and Wainger [27] in their proof of  $L^p$  boundedness of the Hilbert transform. They worked with a distance function  $\rho$ , smooth and positive in  $\mathbb{R}^2 \setminus \{0\}$  which is homogeneous of degree 1 with respect to the dilations (1.6) and considered the analytic family

$$\tilde{H}_\alpha f(x_1, x_2) = \text{p.v.} \int_{-\infty}^{\infty} \rho^\alpha(D) f(x_1 - t, x_2 - t^m) |t|^\alpha \frac{dt}{t}.$$

The result in [27] is that  $\tilde{H}_\alpha$  is bounded on  $L^p$  for  $\alpha < 1 - 1/p$ . Our proof of Theorem 1.3 shows that this result can be improved to  $\tilde{H}_\alpha : L^p \rightarrow L^{p,2}$  if  $\alpha = 1 - 1/p$ ,  $1 < p \leq 2$ .

(iv) The principal value singularity  $\text{p.v. } t^{-1} |t|^{\gamma_1 + \gamma_2 m}$  in the definition of  $H_\gamma$  can be replaced by  $\chi_+^{\gamma_1 + \gamma_2 m - 1} = \lim_{\varepsilon \rightarrow 0} e^{-\varepsilon t} (\Gamma(\gamma_1 + m\gamma_2))^{-1} t_+^{\gamma_1 + m\gamma_2 - 1}$ . This requires only minor changes in the proof of Theorem 1.5.

## 2. Preliminaries

**Notation.** For two quantities  $a$  and  $b$  we write  $a \lesssim b$  or  $b \gtrsim a$  if there exists an absolute positive constant  $C$  so that  $a \leq Cb$ . We shall consistently refer to the homogeneous quasi-norms on Lorentz and Hardy-Lorentz spaces as “norms”, even when the triangle inequality with constant 1 fails. If  $I$  is a (dyadic) cube, then  $x_I$  will denote its center, and  $2^{i_I}$  will denote its side-length. We somewhat abuse notation and use  $2^s I$  to denote the cube with the same center as  $I$  and sidelength  $2^{s+i_I}$ . The Lebesgue measure of a set  $E$  will sometimes be denoted by  $|E|$  and sometimes by  $\operatorname{meas}(E)$ .

**2A. Hardy spaces.** There are many equivalent characterizations of the isotropic Hardy-spaces ([13]), in terms of maximal functions, atomic decompositions and square-functions (see [26] for a rather complete

treatment). We shall use several of them, but most relevant will be the characterization via Littlewood-Paley square-functions, which we choose as a definition.

Let  $\Phi \in \mathcal{S}(\mathbb{R}^n)$  with the property that  $\widehat{\Phi}$  is compactly supported and equal to 1 in a neighborhood of the origin. Let  $\phi_k$  be defined by

$$(2.1) \quad \widehat{\phi}_k(\xi) = \widehat{\Phi}(2^{-k-1}\xi) - \widehat{\Phi}(2^{-k}\xi)$$

Consider the space  $\mathcal{S}'_{\text{restr}}$  of tempered distributions which are *restricted at infinity*; it consists of all  $f \in \mathcal{S}'$  with the property that  $f * \phi \in L^r$  for  $\phi \in \mathcal{S}$ , for sufficiently large  $r < \infty$  (we use the terminology of Stein [26, p.123]). This choice of the test function space allows one to derive versions of the Calderón reproducing formula (*e.g.* one excludes polynomials which have Fourier transforms supported at the origin). For  $0 < p, q < \infty$  we define  $H^{p,q}$  as the space consisting of tempered distributions restricted at  $\infty$  which satisfy

$$(2.2) \quad \|f\|_{H^{p,q}} := \left\| \left( \sum_{k \in \mathbb{Z}} |\phi_k * f|^2 \right)^{1/2} \right\|_{L^{p,q}} < \infty$$

and write  $H^p = H^{p,p}$ . Using arguments in [13], [21] one can show that the definition does not depend on the particular choice of  $\Phi$ . As shown in [21], [31] some aspects in the classical theory simplify by assuming (as we do here) that  $\widehat{\Phi}$  has compact support. In particular for  $b > 0$ ,  $r > 0$  one has the inequality ([21])

$$(2.3) \quad \sup_{|y| \leq 2^{-k}b} |\phi_k * f(x+y)| \leq C_{b,r} (M[|\phi_k * f|^r](x))^{1/r}$$

and (2.3) allows us to take advantage of the Fefferman-Stein theorem concerning  $L^p(\ell^r)$  estimates for the Hardy-Littlewood maximal function  $M$  ([12]). This carries over to Lorentz-spaces. Set

$$S_b f(x) = \left( \sum_{k \in \mathbb{Z}} \sup_{|y| \leq b2^{-k}} |\phi_k * f(x+y)|^2 \right)^{1/2}$$

Since  $\|g\|_{L^{p,q}} \approx \|g^a\|_{L^{p/a,q/a}}^{1/a}$  we obtain that for  $f \in H^{p,q}$

$$(2.4) \quad \|f\|_{H^{p,q}} \approx \|S_b f\|_{L^{p,q}}.$$

The space  $H^{p,q}$  is complete quasi-normed space. We note that the definition can be extended to Hilbert-space valued functions (in fact when proving estimates we may often reduce to finite-dimensional Hilbert spaces with possibly large dimension).

For the purpose of real interpolation consider the Peetre  $K$ -functional  $K(t, f, H^{p_0}, H^{p_1})$ , defined for  $f \in H^{p_0} + H^{p_1}$  as the infimum of  $\|f\|_{H^{p_0}} + t\|f\|_{H^{p_1}}$  over all decompositions  $f = f_0 + f_1$  with  $f_0 \in H^{p_0}$  and  $f_1 \in H^{p_1}$ . Then a straightforward modification of arguments by Jawerth and Torchinsky [19] yields the formula

$$(2.5) \quad K(t, f, H^{p_0}, H^{p_1}) \approx K(t, S_b f, L^{p_0}, L^{p_1}).$$

Consequently, by (2.4) and (2.5) one identifies  $H^{p,q}$  with the real interpolation space  $[H^{p_0}, H^{p_1}]_{\theta,q}$  if  $0 < \theta < 1$  and  $(1-\theta)/p_0 + \theta/p_1 = 1/p$  (see [2]), and the spaces  $H^{p,q}$  can be identified with the spaces in [11], [15] defined by means of various maximal functions or square functions (see [32]).

Let  $\{e_k\}$  be an orthonormal basis of  $\ell^2$ . From standard Hardy space theory [26] we have

$$(2.6) \quad \left\| \sum_k L_k f_k \right\|_{H^{p,q}} \approx \left\| \sum_k \tilde{L}_k f_k e_k \right\|_{L^{p,q}(\ell^2)} = \left\| \left( \sum_k |\tilde{L}_k f_k|^2 \right)^{1/2} \right\|_{L^{p,q}}.$$

where  $L_k$ ,  $\tilde{L}_k$  denote convolution with  $\phi_k$ ,  $\tilde{\phi}_k$ ; here  $\tilde{\phi}_k$  is as above and  $\tilde{\phi}_k = 2^{kd} \tilde{\phi}_0(2^k \cdot)$  so that the Fourier transform of  $\tilde{\phi}$  equals one on the support of  $\widehat{\phi}$ .

Moreover if  $E$  is any finite subset of the integers we have

$$(2.7) \quad \left\| \sum_{k \in E} L_k f_k \right\|_{L^{p,q}} \leq C \left\| \sum_k L_k f_k \right\|_{H^{p,q}}$$

where  $C$  does not depend on  $E$ . Note, however, that convergence in  $L^{p,q}$  may not be compatible with convergence in the sense of tempered distributions, if  $p < 1$  or  $p = 1$ ,  $q > 1$ .

**A Littlewood-Paley decomposition.** It is shown in the classical theory that the above assumptions on  $\Phi$  can be substantially weakened. A general result in this context is in [32]. To eliminate a number of technical error terms in the proof of Theorem 1.1 we shall work with Littlewood-Paley functions localized in space, and in order to have an analogue of the Calderón reproducing formula we will have to use a somewhat unusual version of the Littlewood-Paley decomposition:

**Lemma 2.1.** *Let  $r, N_0$  be nonnegative integers and let  $\varepsilon > 0$ . Then for  $s = 0, \dots, r$  there are radial functions  $\Psi_{(s)}, \psi_{(s)}$  in  $C_0^\infty(\mathbb{R}^d)$  with the following properties.*

(i)  $\Psi_s$  is supported on the ball of radius  $\varepsilon$  centered at the origin, and  $\widehat{\Psi}_s(\xi) - 1 = O(|\xi|^{N_0})$  as  $\xi \rightarrow 0$ . Moreover  $\psi_s = \Psi_s - 2^{-d}\Psi_s(2^{-1}\cdot)$  so that the moments of order  $\leq N_0$  of  $\psi_s$  vanish.

(ii) Define  $\psi_s^k(x) = 2^{kd}\psi_s(2^kx)$  and let  $L_s^k$  be the operator of convolution with  $\psi_s^k$ . Then for every tempered distribution  $f$  restricted at  $\infty$  we have

$$(2.8) \quad f = \sum_{k \in \mathbb{Z}} L_0^k \cdots L_r^k f;$$

moreover if  $S_r^0$  denotes the operator of convolution with  $\Psi_r$  then

$$(2.9) \quad f = S_r^0 f + \sum_{k \geq 1} L_0^k \cdots L_r^k f.$$

The convergence in (2.8), (2.9) holds in the sense of tempered distributions.

**Proof.** Let  $\Psi$  be a radial bump function supported in  $\{x : |x| \leq 2^{-6r-6}\varepsilon\}$  so that  $\widehat{\Psi} - 1 = O(|\xi|^{N+1})$ , and let  $S_0^k$  be the operator of convolution with  $2^{-dk}\Psi(2^{-k}\cdot)$ . Let

$$L_0^k = S_0^k - S_0^{k-1}.$$

We recursively define for  $s = 0, 1, \dots, r-1$

$$(2.10) \quad S_{s+1}^k = (2Id - (S_s^k)^2)(S_s^k)^2$$

$$(2.11) \quad L_{s+1}^k = (2Id - (S_s^k)^2 - (S_s^{k-1})^2)(S_s^k + S_s^{k-1})$$

and note the identity

$$(2.12) \quad S_{s+1}^k - S_{s+1}^{k-1} = (S_s^k - S_s^{k-1})L_{s+1}^k$$

so that  $S_{s+1}^k - S_{s+1}^{k-1} = L_0^k \cdots L_{s+1}^k$ . One can check inductively that each  $S_s^k$  is the operator of convolution with  $2^{kd}\Psi^s(2^k\cdot)$  where the radial bump function  $\Psi^s$  is supported in  $\{x : |x| \leq 2^{-6(r-s+1)}\varepsilon\}$  and  $\widehat{\Psi}^s(\xi) - 1 = O(|\xi|^{N_0+1})$  as  $\xi \rightarrow 0$ , and that the operators  $L_s^k, S_s^0$  have all the desired properties.  $\square$

*Remark.* We note that (2.6) holds if  $L_k, \tilde{L}_k$  are replaced by any of the operators  $L_s^k$  above, or perhaps by a composition of finitely many such operators. This remark holds under the condition that the number  $N_0$  of vanishing moments is sufficiently large (in dependence of  $p$ ; specifically we need  $N_0 \geq n(1/p - 1)$ ).

**Parabolic dilations.** One may define Hardy spaces with respect to a nonisotropic dilation structure [3]. In this paper we need to consider such Hardy-spaces on  $\mathbb{R}^2$  defined with respect to the scaling  $(x_1, x_2) \mapsto (tx_1, t^m x_2)$ , for a fixed real number  $m > 1$ .

If we redefine the function  $\phi_k$  to be  $\widehat{\phi}_k(\xi_1, \xi_2) = \widehat{\Phi}(2^{-(k+1)}\xi_1, 2^{-(k+1)m}\xi_2) - \widehat{\Phi}(2^{-k}\xi_1, 2^{-km}\xi_2)$  then the operator of convolution with  $\phi_k$  is a Littlewood-Paley projection to the region  $|\xi_1| + |\xi_2|^{1/m} \sim 2^k$ . We may then define  $H_{parabolic}^p$  as the space of distributions  $f$  restricted at  $\infty$ , for which  $\|(\sum_k |\widehat{\phi}_k * f|^2)^{1/2}\|_p$  is finite. Similarly one can define parabolic Hardy-Lorentz space and the obvious analogues of the statements in the previous subsections remain true.

**Product type Hardy spaces.** Let  $\{L_{k_1, k_2}\}_{k_1, k_2 \in \mathbb{Z}}$  be a product Littlewood-Paley decomposition on  $\mathbb{R}^2$ , where  $L_{k_1, k_2}$  is a multiplier with symbol supported in the region  $\{(\xi_1, \xi_2) : |\xi_1| \sim 2^{k_1}, |\xi_2| \sim 2^{k_2}\}$ ; we may assume that  $L_{k_1, k_2}$  is the operator of convolution with  $\phi_{k_1} \otimes \phi_{k_2}$  where  $\phi_{k_1}, \phi_{k_2}$  are as above (defined on the real line).

If  $0 < p, q < \infty$ , we define the product Hardy-Lorentz space  $H_{prod}^{p,q}$  to be the quasi-Banach space which consists of all tempered distributions restricted at  $\infty$  for which

$$\|f\|_{H_{prod}^{p,q}} = \left\| \left( \sum_{k_1} \sum_{k_2} |L_{k_1, k_2} f|^2 \right)^{1/2} \right\|_{L^{p,q}}$$

is finite. We define  $H_{prod}^p$  to be  $H_{prod}^{p,p}$ .

The formulas for interpolation of Hardy-Lorentz-spaces remain true; in fact (2.5) was proved in this context in [19]. Moreover analogues of (2.6), (2.7) remain true for the operators  $L_{k_1, k_2}$ . These can be proved by using the theory of product-type singular integral operators (see e.g. [6], [14]).

**2B. Analytic interpolation in Lorentz spaces.** We need a version of a theorem by Sagher [22] concerning analytic families of operators acting on Lorentz spaces. It has been observed in [23] and [16] that Sagher's arguments carry over to somewhat more general situations; we now recall the version which appeared in [16].

We denote by  $S$  the strip  $S = \{z : 0 < \operatorname{Re}(z) < 1\}$  and by  $\overline{S}$  its closure. A function  $g$  on  $\overline{S}$  is said to be of *admissible growth* if there is  $a < \pi$  so that  $|g(z)| \lesssim \exp(e^{a|\operatorname{Im}(z)|})$  for  $z \in \overline{S}$ . Let  $X_0$  and  $X_1$  be two Banach spaces, compatible in the sense of interpolation theory, and assume that there is a subspace  $W$  of  $X_0 \cap X_1$  which is dense in both  $X_0$  and  $X_1$ . For  $z \in \overline{S}$  let  $\mathcal{T}_z$  be an operator which maps functions in  $W$  to measurable functions on  $\mathbb{R}^n$ ;  $\mathcal{T}_z$  is then called an analytic family if for any  $f \in W$  and almost every  $x \in \mathbb{R}^n$  the function  $z \rightarrow \mathcal{T}_z f(x)$  is analytic in  $S$  and continuous and of admissible growth in  $\overline{S}$ . Now if

$$(2.13) \quad \|\mathcal{T}_z f\|_{L^{p_i, q_i}} \leq C_i(z) \|f\|_{X_i}, \quad i = 0, 1,$$

and if  $C_i(z)$  is of admissible growth then the result in [16] states that  $T_\theta$  maps the complex interpolation space  $[X_0, X_1]_\theta$  boundedly to  $L^{p_\theta, q_\theta}$ ; here  $(1/p_\theta, 1/q_\theta) = (1-\theta)(1/p_0, 1/q_0) + \theta(1/p_1, 1/q_1)$ . We shall need the following consequence of this result.

**Lemma 2.2.** For  $k \in \mathbb{Z}$  and  $z \in S$  let  $T_{k,z}$  be an operator which maps functions in  $W$  to measurable functions on  $\mathbb{R}^n$  and assume that  $T_{k,z}$  is an analytic family, for each  $k$ . Suppose that for all  $f \in W$

$$(2.14) \quad \left\| \sum_{k \in E} |T_{k, i\tau} f| \right\|_{L^1} \leq C(i\tau) \|f\|_{X_0}$$

$$(2.15) \quad \left\| \sup_{k \in E} |T_{k, 1+i\tau} f| \right\|_{L^{1,\infty}} \leq C(1+i\tau) \|f\|_{X_1}$$

for any finite subset  $E \subset \mathbb{Z}$ , with admissible constants  $C(i\tau)$ ,  $C(1+i\tau)$ . Let  $0 < \theta < 1$ . Then

$$(2.16) \quad \left\| \left( \sum_{k \in \mathbb{Z}} |T_{k, \theta} f|^q \right)^{1/q} \right\|_{L^{1,q}} \lesssim \|f\|_{[X_0, X_1]_\theta}$$

if  $1/q_\theta = 1 - \theta$ .

**Proof.** Fix  $\tilde{f} \in [X_0, X_1]_\theta$  and  $E \subset \mathbb{Z}$  be finite. There are measurable functions  $g_k$  such that  $\sum |g_k(x)|^{q'} \leq 1$  and

$$\left| \sum_{k \in E} T_{k, \theta} \tilde{f}(x) g_k(x) \right| \geq \frac{1}{2} \left( \sum_{k \in E} |T_{k, \theta} \tilde{f}(x)|^q \right)^{1/q}$$

for almost every  $x \in \mathbb{R}^n$ . Define  $g_{k,z}(x) = \frac{g_k(x)}{|g_k(x)|} |g_k(x)|^{q'z}$  if  $g_k(x) \neq 0$ , and  $g_{k,z}(x) = 0$  if  $g_k(x) = 0$ .

Now define an analytic family by  $\mathcal{T}_z f(x) = \sum_{k \in E} T_{k,z} f(x) g_{k,z}(x)$ . Then the assumptions (2.14-15) imply the boundedness of  $\mathcal{T}_{i\tau}$  from  $X_0$  to  $L^1$  and of  $\mathcal{T}_{1+i\tau}$  from  $X_1$  to  $L^{1,\infty}$ , with admissible constants. One deduces the boundedness of  $\mathcal{T}_\theta$  from  $[X_0, X_1]_\theta$  to  $L^{1,q}$ . The constants are independent of  $E$  and the choice of  $\{g_k\}$ . This implies

$$\left\| \left( \sum_{k \in E} |T_{k,\theta} \tilde{f}|^q \right)^{1/q} \right\|_{L^{1,q}} \leq C \|\tilde{f}\|_{[X_0, X_1]_\theta}$$

with  $C$  being independent of  $E$  and  $\tilde{f}$ . The finiteness assumption on  $E$  can be removed by applications of the monotone convergence theorem.  $\square$

**2C. A vector-valued inequality.** We shall use the following observation which can serve as an elementary substitute for the failing  $L^p(\ell^1)$  inequality for the vector-valued Hardy-Littlewood maximal operator ([12]). It is just the dual version of a scalar maximal inequality.

**Lemma 2.3.** *Let  $\Phi \in L^1(\mathbb{R}^d)$  so that for each  $\theta \in S^{d-1}$  the function  $r \mapsto |\Phi(r\theta)|$  is decreasing in  $r > 0$ . Let  $\{t_k\}_{k \in \mathbb{Z}}$  be a collection of positive numbers and let  $P_k$  be the operator of convolution with  $t_k^d \Phi(t_k \cdot)$ . Then for  $1 \leq p < \infty$*

$$(2.17) \quad \left\| \sum_k |P_k f_k| \right\|_p \leq C_p \|\Phi\|_1 \left\| \sum_k |f_k| \right\|_p.$$

**Proof.** We may assume that  $\Phi$  is nonnegative. Then by duality the assertion follows immediately from the  $L^{p'}$  boundedness of the maximal operator  $w \mapsto \sup_k |P_k w|$ ; the latter is a consequence of the method of rotation and the bounds for the one-dimensional Hardy-Littlewood operator (see [26, p.72-73]).  $\square$

**2D. Averaging functions in  $L^{1,q}$ .** The triangle inequality fails in  $L^{1,q}$  if  $q > 1$ , but the following Lemma, proved for  $q = \infty$  by Stein and N. Weiss [28], can often serve as a substitute. For  $1 < q < \infty$  the statement follows from the cases  $q = 1$  and  $q = \infty$  by interpolation.

**Lemma 2.4.** *Suppose that  $\|f_i\|_{L^{1,q}} \leq 1$  and  $\sum |c_i| \leq 1$ . Then*

$$\left\| \sum_i c_i f_i \right\|_{L^{1,q}} \lesssim \sum_i |c_i| (1 + \log_+ |c_i|)^{1-\frac{1}{q}}.$$

### 3. A stopping time construction

We shall use an abstract form of the Calderón-Zygmund decomposition, in which no nesting or doubling properties are assumed. The argument is related to the stopping time construction in [7].

**Lemma 3.1.** *Let  $\preceq, \subseteq$  be partial orders on a set  $\Lambda$ ; we also use the notation  $\prec$  synonymously with  $\not\preceq$ . Let  $\Gamma$  be a finite subset of  $\Lambda$ , let  $\nu$  be a non-negative measure on  $\Gamma$ , and let  $A : \Lambda \rightarrow \mathbb{R}^+$  be a positive function.*

*Assume that for each  $\gamma \in \Gamma$  and  $N > 0$  the set*

$$(3.1) \quad \{\lambda \in \Lambda : A(\lambda) \leq N \text{ and } \gamma \subseteq \lambda\}$$

*is finite.*

*Then one can find a subset  $\mathcal{B}$  of  $\Lambda$  and a map  $q : \Gamma \rightarrow \Lambda$  which have the following properties.*

- (1)  $\gamma \subseteq q(\gamma)$  for all  $\gamma \in \Gamma$ .
- (2) If  $q(\gamma) \notin \mathcal{B}$  then  $q(\gamma) = \gamma$ .

$$(3) \quad \sum_{\lambda \in \mathcal{B}} A(\lambda) \leq \nu(\Gamma)$$

(4) For all  $\lambda \in \Lambda$ , we have

$$\nu(\{\gamma \in \Gamma : q(\gamma) \prec \lambda, \gamma \subseteq \lambda\}) < A(\lambda).$$

**Proof.** Define

$$(3.2) \quad \Lambda_* = \Gamma \cup \{\lambda \in \Lambda : A(\lambda) \leq \nu(\Gamma) \text{ and } \gamma \subseteq \lambda \text{ for some } \gamma \in \Gamma\}$$

By the finiteness of  $\Gamma$  and the finiteness assumption on the sets (3.1) the set  $\Lambda_*$  is finite. Suppose we have found  $q$  and  $\mathcal{B}$  with properties (1)-(4) relatively to  $\Lambda_*$  then (1)-(4) are unchanged if  $\Lambda_*$  is enlarged to  $\Lambda$ . Hence it suffices to give a proof under the additional assumption that  $\Lambda$  is finite.

We now induct on the cardinality of  $\Lambda$ . The lemma is vacuously true when  $\Lambda$  is empty, with  $\mathcal{B}$  being empty and  $q$  being the empty function.

Now suppose inductively that  $\Lambda$  is non-empty, and that the lemma is true for all sets  $\Lambda$  of lesser cardinality. Choose an element  $\lambda_{max} \in \Lambda$  which is maximal with respect to the partial ordering  $\preceq$ , and let  $\Lambda' = \Lambda - \{\lambda_{max}\}$ . Define the set  $\Gamma' \subset \Gamma$  by

$$\Gamma' = \Gamma \cap \Lambda'$$

if the estimate

$$(3.3) \quad \nu(\{\gamma \in \Gamma : \gamma \subseteq \lambda_{max}\}) < A(\lambda_{max})$$

holds, and by

$$\Gamma' = \{\gamma \in \Gamma : \gamma \not\subseteq \lambda_{max}\}$$

otherwise.

Now apply the induction hypothesis with  $\Lambda$  replaced by  $\Lambda'$ ,  $\Gamma$  replaced by  $\Gamma'$ , and  $A$  and  $\nu$  replaced by their restrictions to  $\Lambda'$  and  $\Gamma'$  respectively. This gives us a set  $\mathcal{B}' \subset \Lambda'$  and an assignment  $q' : \Gamma' \rightarrow \Lambda'$  satisfying analogues (1')-(4') of the desired properties (1)-(4).

Define the subset  $\mathcal{B}$  of  $\Lambda$  by  $\mathcal{B} = \mathcal{B}'$  if (3.3) holds, and  $\mathcal{B} = \mathcal{B}' \cup \{\lambda_{max}\}$  if (3.3) fails. Define  $q : \Gamma \rightarrow \Lambda$  by setting  $q(\gamma) = q'(\gamma)$  if  $\gamma \in \Gamma'$ , and  $q(\gamma) = \lambda_{max}$  if  $\gamma \in \Gamma \setminus \Gamma'$ .

We now claim that (1)-(4) holds for these choices of  $\mathcal{B}$  and  $q$ . The claims (1), (2) are easily verified from (1'), (2'), and the construction of  $\mathcal{B}$  and  $q$ . If (3.3) holds then  $\mathcal{B} = \mathcal{B}'$  and (3) follows from (3'); otherwise,  $\mathcal{B} = \mathcal{B}' \cup \{\lambda_{max}\}$  and (3) follows from (3'), the construction of  $\Gamma'$ , and the failure of (3.3).

It remains to verify (4). First suppose that  $\lambda \neq \lambda_{max}$ , so that  $\lambda \in \Lambda'$ . Then (4) follows from (4'), because the elements  $\gamma$  of  $\Gamma \setminus \Gamma'$  satisfy  $q(\gamma) = \lambda_{max}$  and thus cannot contribute to the left-hand side of (4) by the maximality of  $\lambda_{max}$ .

Now suppose that  $\lambda = \lambda_{max}$ . If (3.3) holds, then (4) is immediate. If (3.3) fails, then by construction the left-hand side of (4) is zero. Thus (4) holds in all cases, and the induction step is complete.  $\square$

We remark that the finiteness assumption (3.1) may be dropped if one is willing to replace the induction by transfinite induction (*i.e.* use Zorn's lemma). One can then prove this lemma for arbitrary  $\Lambda$ .

#### 4. Integrals along plane curves

In this and the next section we shall always be working in the plane  $\mathbb{R}^2$ . We fix a real number  $m > 1$ , all constants may implicitly depend on  $m$ . We define  $H_{\text{parabolic}}^1$  to be the one-parameter Hardy space with respect to the scaling (1.6).

The proofs of our results concerning plane curves are based on the following key estimate.

**Proposition 4.1.** *For each integer  $l$  let  $\eta_l$  be a  $C^\infty$  function with compact support in  $[1/2, 2]$  or in  $[-2, -1/2]$ , with  $C^4$  norms uniformly bounded in  $l$ .*

Let  $d\mu_l$  be the measure defined by

$$\int f d\mu_l = \int f(x_1 - t, x_2 - |t|^m) 2^l \eta_l(2^l t) dt.$$

Then for any vector-valued function  $F = \{f_l\}_{l \in \mathbb{Z}}$ ,

$$(4.1) \quad \left\| \left( \sum_l |f_l * d\mu_l|^2 \right)^{1/2} \right\|_{L^{1,2}} \lesssim \|f\|_{H_{\text{parabolic}}^1(\ell^2)}.$$

We allow the  $f_l$  themselves to be Hilbert space valued functions, and  $|\cdot|$  is then to be interpreted as the Hilbert space norm.

In the next section, we shall see how this proposition implies  $L^{1,2}$  and  $L^{p,2}$  mapping properties for the Hilbert transform on plane curves and similar objects; this will be done by exploiting the fact that the  $d\mu_l$  have essentially disjoint frequency supports if some moment conditions are assumed on the  $\eta_l$ . The estimate (4.1) should be compared with the bound

$$\left\| \sup_l |f * d\mu_l| \right\|_{L^{1,\infty}} \lesssim \|f\|_{H_{\text{parabolic}}^1}$$

proven in Christ [7]. Our techniques shall be closely related to those in that paper.

**Proof.** We may decompose  $f$  atomically as  $f = \sum_I c_I P_I(b_I)$ , where the  $I$  are  $2^k \times 2^{mk+\vartheta}$  rectangles with sides parallel to the axes, and  $k, km + \vartheta$  are integers,  $0 \leq \vartheta < 1$ . The  $c_I$  are non-negative numbers such that  $\sum_I c_I \sim \|f\|_{H_{\text{parabolic}}^1(\ell^2)}$ , the  $b_I$  satisfy  $\|b_I\|_{L^2(\ell^2)} \lesssim |I|^{-1/2}$ , and  $P_I$  is the projection operator defined by

$$P_I[b](x) = \left( b(x) - \frac{1}{|I|} \int_I b(x) dx \right) \chi_I(x).$$

Note that the definition of  $P_I$  makes sense as acting on scalar valued functions or on vector-valued functions, as above. By the translation trick in [7] (attributed to P. Jones) we may assume that the cubes  $I$  are dyadic. Henceforth we shall refer to the  $I$  as (parabolic) cubes. It thus suffices to show the estimate

$$(4.2) \quad \left\| \left( \sum_l \left| \sum_I c_I P_I[b_{I,l}] * d\mu_l \right|^2 \right)^{1/2} \right\|_{L^{1,2}} \lesssim (\sum_I c_I) (\sup_I |I|^{1/2}) \left\| \left( \sum_l |b_{I,l}|^2 \right)^{1/2} \right\|_2$$

for arbitrary collections  $I$  of cubes, non-negative numbers  $c_I$ , and arbitrary measurable functions  $b_{I,l}$ . By limiting arguments it is sufficient to prove the analogue of (4.2), where the sums in  $l$  and the sums in  $I$  are extended over finite sets (with bounds independent of the cardinalities). Henceforth we make this finiteness assumption.

Fix the  $I$  and  $c_I$ . By complex interpolation (Lemma 2.2) it suffices to show that

$$(4.3) \quad \left\| \left( \sum_l \left| \sum_I c_I P_I[b_{I,l}] * d\mu_l \right|^q \right)^{1/q} \right\|_{L^{1,q}} \lesssim (\sum_I c_I) (\sup_I |I|^{1/q'}) \left\| \left( \sum_l |b_{I,l}|^q \right)^{1/q} \right\|_q$$

holds for  $q = 1$  and  $q = \infty$  and all (complex) functions  $b_{I,l}$ .

When  $q = 1$ , (4.3) simplifies to

$$\sum_l \sum_I c_I \|P_I[b_{I,l}] * d\mu_l\|_1 \lesssim \left(\sum_I c_I\right) \sup_I \sum_l \|b_{I,l}\|_1,$$

and the claim follows from Young's inequality, the finite mass of  $d\mu_l$ , and the fact that  $P_I$  is bounded on  $L^1$ . Thus it remains to prove the  $q = \infty$  endpoint, namely

$$\left\| \sup_l \left| \sum_I c_I P_I[b_{I,l}] * d\mu_l \right| \right\|_{L^{1,\infty}} \lesssim \left(\sum_I c_I\right) \sup_I \sup_l |I| \|b_{I,l}\|_\infty.$$

We may assume that

$$(4.4) \quad \sup_I \sup_l |I| \|b_{I,l}\|_\infty \leq 1$$

Writing  $a_{I,l} = P_I[b_{I,l}]$ , we thus see that  $a_{I,l}$  is supported on  $I$ , has mean zero, and  $\|a_{I,l}\|_\infty \lesssim |I|^{-1}$ , and our task is now to show that

$$(4.5) \quad \text{meas}(\{\sup_l \left| \sum_I c_I a_{I,l} * d\mu_l \right| \gtrsim \alpha\}) \lesssim \alpha^{-1} \sum_I c_I$$

for all  $\alpha > 0$ .

Fix  $\alpha > 0$ . We shall use a sort of Calderón-Zygmund decomposition and will first look at the “good” cubes contributing to a function which is  $O(\alpha)$ . Let  $\mathcal{G}$  be the family of all  $I$  for which

$$(4.6) \quad M\left(\sum_{I'} c_{I'} \frac{\chi_{I'}}{|I'|}\right)(x) \leq \alpha \text{ for some } x \in I;$$

here  $M$  is the Hardy-Littlewood maximal operator with respect to the scaling (1.6).

We consider the contribution of the cubes in  $\mathcal{G}$  to (4.5). The  $L^\infty$  norm of  $\sum_{I \in \mathcal{G}} c_I \frac{\chi_I}{|I|}$  is  $O(\alpha)$ , to see this, consider for each  $x_0$  the smallest cube in  $\mathcal{G}$  containing  $x_0$  and apply (4.6) for this cube. We now apply Chebyshev's inequality and the standard fact [28] that the maximal function associated to the curve  $(t, |t|^m)$  is bounded on  $L^2$ . This yields

$$(4.7) \quad \begin{aligned} & \text{meas}\left(\left\{x : \sup_l \left| \sum_{I \in \mathcal{G}} c_I a_{I,l} * d\mu_l \right| \geq \alpha\right\}\right) \\ & \leq \alpha^{-2} \left\| \sup_l \left| \sum_{I \in \mathcal{G}} c_I a_{I,l} * d\mu_l \right| \right\|_2^2 \lesssim \alpha^{-2} \left\| \sup_l \sum_{I \in \mathcal{G}} c_I \frac{\chi_I}{|I|} * |d\mu_l| \right\|_2^2 \\ & \lesssim \alpha^{-2} \left\| \sum_{I \in \mathcal{G}} c_I \frac{\chi_I}{|I|} \right\|_2^2 \lesssim \alpha^{-1} \left\| \sum_{I \in \mathcal{G}} c_I \frac{\chi_I}{|I|} \right\|_1 \lesssim \alpha^{-1} \sum_I c_I. \end{aligned}$$

Thus we may restrict our attention to the “bad” cubes. By the Hardy-Littlewood inequality, the  $L^{1,\infty}$  norm of  $M(\sum_I c_I \chi_I / |I|)$  is  $O(\sum_I c_I)$ , and so by the definition of  $\mathcal{G}$

$$\text{meas}(\cup_{I \notin \mathcal{G}} I) \lesssim \alpha^{-1} \sum_I c_I.$$

Let  $C > 1$  and  $CI$  denote the cube expanded by  $C$  (with same center as  $I$ ). By the Hardy-Littlewood inequality again we have

$$(4.8) \quad \text{meas}(\cup_{I \notin \mathcal{G}} CI) \lesssim \alpha^{-1} \sum_I c_I.$$

To complete the proof of (4.5) we shall prove the stronger square-function estimate

$$(4.9) \quad \text{meas}\left(\left\{x : \left(\sum_l \left|\sum_{I \notin \mathcal{G}} c_I a_{I,l} * d\mu_l(x)\right|^2\right)^{1/2} \geq \alpha\right\}\right) \lesssim \alpha^{-1} \sum_I c_I.$$

In order to prove (4.9) we use an abstract version of the Calderón-Zygmund decomposition based on Lemma 3.1. We first describe the sets  $\Lambda$  and  $\Gamma$  which occur in this lemma. If  $m \geq 2$  we define  $\Lambda$  as the set of all dyadic rectangles  $Q$  of dimensions  $2^\sigma \times 2^{\sigma+(m-1)\tau+\vartheta}$  for integers  $\sigma, \tau$  and for  $\vartheta \in [0, 1)$ , where  $\sigma \leq \tau$ , and  $(m-1)\tau + \vartheta$  is the smallest integer  $\geq (m-1)\tau$  (*i.e.*  $\vartheta = 0$  if  $m$  is an integer). Note that  $\sigma, \tau$  and  $\vartheta$  are unique for each  $Q$  and we shall write  $\sigma = \sigma(Q), \tau = \tau(Q), \vartheta = \vartheta(Q)$ . If  $1 < m < 2$  we define  $\Lambda$  similarly, with the additional requirement that we only admit those  $\tau$  for which the fractional part of  $(m-1)(\tau - \sigma)$  is  $< m-1$ ; this is to ensure that  $\tau(Q)$  is well defined. In both cases the subset  $\Gamma$  is the set of parabolic cubes  $I$  for which  $c_I \neq 0$  and which do not belong to  $\mathcal{G}$ ; by assumption  $\Gamma$  is finite. Note that one has  $\tau(I) = \sigma(I)$  for parabolic cubes  $I$ .

We wish to partially order the set  $\Lambda$  by requiring  $Q \prec Q'$  if  $\tau(Q) < \tau(Q')$ ; note that then  $Q$  and  $Q'$  are incomparable under  $\preceq$  if  $\tau(Q) = \tau(Q')$  and  $Q \neq Q'$ . Finally we take set inclusion  $\subseteq$  as the second partial order in Lemma 3.1.

We define the tendril  $T(Q)$  to be the set

$$(4.10) \quad T(Q) = \{x + (t, |t|^m) : x \in 2Q, |t| \leq 2^{\tau(Q)+2}\}.$$

Note that  $|T(Q)| \sim 2^{\sigma(Q)+m\tau(Q)} + 2^{2\sigma(Q)+(m-1)\tau(Q)}$  for any rectangle  $Q$  parallel to the axes, and therefore

$$(4.11) \quad |T(Q)| \sim 2^{\sigma(Q)+m\tau(Q)} \quad \text{for } Q \in \Lambda.$$

The function  $A(Q)$  in Lemma 3.1 is then defined by

$$A(Q) = \alpha 2^{\sigma(Q)+m\tau(Q)},$$

and the measure  $\nu$  is defined by

$$\nu(\{I\}) = c_I.$$

The finiteness condition in the proof of Lemma 3.1 is easily verified and we find a map  $I \mapsto q(I)$  defined on  $\Gamma$  so that  $I \subseteq q(I)$  and

$$(4.12) \quad \sum_{\substack{I \in \Gamma \\ q(I) \prec Q \\ I \subseteq Q}} c_I < \alpha |T(Q)|$$

for all  $Q \in \Lambda$ , and

$$\text{meas}(\cup_{I \in \Gamma} T(q(I))) \lesssim \frac{1}{\alpha} \sum_I c_I + \text{meas}(\cup_{I \in \Gamma} T(I));$$

the latter inequality follows from statements (2) (3), (4) of Lemma 3.1. Since for parabolic cubes  $I$  the tendril  $T(I)$  is contained in a fixed dilate of  $I$  and since  $\Gamma \cap \mathcal{G} = \emptyset$  one has actually

$$(4.13) \quad \text{meas}(\cup_{I \in \Gamma} T(q(I))) \lesssim \frac{1}{\alpha} \sum_I c_I,$$

by (4.8).

For any  $I, l$  we see that  $a_{I,l} * d\mu_l$  is supported in  $T(q(I))$  if  $l < \tau(q(I))$ . In view of (4.13) the inequality (4.9) follows from

$$\text{meas}\left(\left\{x : \left(\sum_l \left|\sum_{I:l \geq \tau(q(I))} c_I a_{I,l} * d\mu_l\right|^2\right)^{1/2} \geq \alpha\right\}\right) \lesssim \alpha^{-1} \sum_I c_I.$$

It suffices by Chebyshev's inequality to prove the  $L^2$  estimate

$$(4.14) \quad \left\| \left( \sum_l \left| \sum_{\substack{I \in \Gamma \\ l \geq \tau(q(I))}} c_I a_{I,l} * d\mu_l \right|^2 \right)^{1/2} \right\|_2^2 \lesssim \alpha \sum_I c_I.$$

Let

$$(4.15) \quad \Gamma(m) = \{I \in \Gamma : \tau(q(I)) = m\}.$$

By the triangle inequality it suffices to show

$$\left\| \left( \sum_l \left| \sum_{I \in \Gamma(l-s)} c_I a_{I,l} * d\mu_l \right|^2 \right)^{1/2} \right\|_2^2 \lesssim 2^{-s} \alpha \sum_I c_I$$

for all  $s \geq 0$ .

Fix  $s$ . It then suffices to show that for each  $l$

$$(4.16) \quad \left\| \sum_{I \in \Gamma(l-s)} c_I a_{I,l} * d\mu_l \right\|_2^2 \lesssim 2^{-s} \alpha \sum_{I \in \Gamma(l-s)} c_I$$

for each  $l$ , since the claim follows by summing in  $l$ . By scaling (with respect to the parabolic dilations (1.6) and taking into account the definition of  $\tau(Q)$ ) we see that it suffices to prove (4.16) for  $l = 0$ . Expanding the left-hand side of (4.16), we reduce to

$$\sum_{I, I' \in \Gamma(-s)} c_I c_{I'} |\langle a_{I,0} * d\mu_0, a_{I',0} * d\mu_0 \rangle| \lesssim 2^{-s} \alpha \sum_{I \in \Gamma(-s)} c_I.$$

By symmetry we may assume that  $|I'| \leq |I|$ . It then suffices to show that

$$(4.17) \quad \sum_{\substack{I' \in \Gamma(-s) \\ |I'| \leq |I|}} c_{I'} |\langle a_{I,0} * d\mu_0, a_{I',0} * d\mu_0 \rangle| \lesssim 2^{-s} \alpha,$$

for all  $I \in \Gamma(-s)$ .

Fix  $I \in \Gamma(-s)$  with center  $x_I$ .  $I$  has dimension  $2^{\tau(I)} \times 2^{m\tau(I)+\vartheta(I)}$ ; since  $I \subseteq q(I)$  by Lemma 3.1, (1), and  $\sigma(q(I)) \leq \tau(q(I))$  by definition of  $\Lambda$  we see that

$$(4.18) \quad \tau(I) \leq \tau(q(I)) = -s.$$

Rewrite the left-hand side of (4.12) as

$$(4.19) \quad \sum_{I': |I'| \leq |I|, \tau(q(I')) = -s} c_{I'} |\langle a_{I,0} * F, a_{I',0} \rangle|$$

where  $F = d\mu_0 * \widetilde{d\mu}_0$  (and  $\sim$  refers to reflection in the argument). Observe that  $F$  is supported on a sector

$$\{(x_1, x_2) : |x_2| \lesssim |x_1|\}$$

and obeys the estimates

$$|\nabla^\alpha F(x)| \lesssim |x|^{-1-|\alpha|}$$

for all multiindices  $\alpha$  with  $|\alpha| \leq 2$ . From the size conditions on  $a_{I,0}$ , this implies

$$|a_{I,0} * F(x)| \lesssim 2^{-\tau(I)}$$

and by the moment conditions on  $a_{I,0}$

$$|\nabla^\alpha(a_{I,0} * F)(x)| \lesssim 2^{\tau(I)}|x - x_I|^{-2-|\alpha|}, \quad \text{if } |x - x_I| \geq 2^{\tau(I)+1}, |\alpha| \leq 1.$$

This in turn implies from the size and moment conditions on  $a_{I',0}$  and the assumption  $|I'| \leq |I|$  that

$$|\langle a_{I,0} * F, a_{I',0} \rangle| \lesssim 2^{2\tau(I)} \text{diam}(I \cup I')^{-3},$$

where the diameter is respect to the Euclidean metric.

Thus it suffices to show that

$$(4.20) \quad \sum_{\substack{I' \in \Gamma(-s) \\ |I'| \leq |I|}} c_{I'} \text{diam}(I \cup I')^{-3} \lesssim 2^{-2\tau(I)} 2^{-s} \alpha.$$

For  $\sigma \leq -s$ , let  $\mathcal{R}_{\sigma,s}$  be the set of dyadic rectangles of dimensions  $(2^\sigma, 2^{\sigma-(m-1)(s-1)+\vartheta})$  so that  $0 \leq \vartheta < 1$ . Observe that  $\mathcal{R}_{\sigma,s}$  is a subset of  $\Lambda$  consisting of rectangles  $R$  with  $\tau(R) = -s+1$ . Also let  $\mathcal{W}_\sigma$  be the set of isotropic dyadic cubes of dimensions  $(2^\sigma, 2^\sigma)$ ; then each  $W \in \mathcal{W}_\sigma$  is a union of  $\sim 2^{(m-1)(s-1)}$  rectangles in  $\mathcal{R}_{\sigma,s}$ , with disjoint interiors.

If  $I' \in \Gamma(-s)$  with  $|I'| \leq |I|$  then  $I'$  has dimensions  $(2^{\sigma(I')}, 2^{\sigma(I')-(m-1)s})$  and  $\sigma(I') \leq \sigma(I) = \tau(I)$ , and therefore every such  $I'$  is contained in a unique rectangle  $R \in \mathcal{R}_{\tau(I),s}$ . Since  $\tau(q(I')) = -s$  and  $\tau(R) = -s+1$  we have from Lemma 3.1, (4),

$$\sum_{\substack{I' \in \Gamma(-s) \\ |I'| \leq |I| \\ I' \subseteq R}} c_{I'} \lesssim \alpha |T(R)| \lesssim \alpha 2^{\tau(I)-ms}$$

and therefore

$$\begin{aligned} & \sum_{\substack{I' \in \Gamma(-s) \\ |I'| \leq |I|}} c_{I'} \text{diam}(I \cup I')^{-3} \\ &= \sum_{W \in \mathcal{W}_{\tau(I)}} \sum_{\substack{R \in \mathcal{R}_{\tau(I),s} \\ R \subset W}} \sum_{\substack{I' \in \Gamma(-s) \\ |I'| \leq |I| \\ I' \subset R}} c_{I'} \text{diam}(I \cup I')^{-3} \\ &\lesssim \alpha 2^{\tau(I)-ms} \sum_{W \in \mathcal{W}_{\tau(I)}} (2^{\tau(I)} + \text{dist}(W, I))^{-3} \text{card}(\{R \in \mathcal{R}_{\tau(I),s} : R \subset W\}) \\ &\lesssim \alpha 2^{\tau(I)-s} \sum_{W \in \mathcal{W}_{\tau(I)}} (2^{\tau(I)} + \text{dist}(W, I))^{-3} \lesssim 2^{-2\tau(I)} \alpha 2^{-s} \end{aligned}$$

which is (4.20).  $\square$

## 5. Integrals along plane curves, cont.

We now prove Theorems 1.4 and 1.5. Following [5] we work with an angular Littlewood-Paley decomposition.

Let  $\zeta \in C_0^\infty(\mathbb{R}_+)$  so that  $\zeta(s) = 1$  if  $s \in ((10^m m)^{-1}, 10^m m)$  and define  $Q_l$  by

$$(5.1) \quad \widehat{Q_l f}(\xi) = q_l(\xi) \widehat{f}(\xi) = \zeta(2^{l(m-1)} |\xi_1| / |\xi_2|) \widehat{f}(\xi).$$

The operators  $Q_l$  form a Littlewood-Paley family of multipliers supported in sectors. Note that  $q_l(\xi) = 1$  whenever  $\xi$  is normal to the curves  $(t, \pm |t|^m)$  if  $2^{l-1} \leq |t| \leq 2^{l+1}$ .

Let  $\chi_0$  be a smooth and even function on  $\mathbb{R}$  so that  $\chi_0(s) = 1$  if  $|s| \leq 1/2$  and  $\chi_0(s) = 0$  if  $|s| \geq 1$ . Define  $\mathcal{P}_l$  by  $\widehat{\mathcal{P}_l f}(\xi) = \chi_0(|(2^{-l}\xi_1, 2^{-l m}\xi_2)|) \widehat{f}(\xi)$ .

Observe that the multiplier  $q_l$  satisfies the estimates  $\partial^\alpha q_l(\xi) = O(|\xi_1|^{-\alpha_1} |\xi_2|^{-\alpha_2})$  uniformly in  $l$ . Therefore by standard product theory we have the estimate

$$(5.2) \quad \| \{(Id - \mathcal{P}_l) Q_l f\} \|_{H_{prod}^1(\ell^2)} \lesssim \| \{Q_l f\} \|_{H_{prod}^1(\ell^2)} \lesssim \| f \|_{H_{prod}^1}$$

where  $f$  itself may be a Hilbert-space valued function.

We now consider the maximal function  $Mf$ . We show that

$$(5.3) \quad \| \sup_l |d\mu_l * f| \|_{L^{1,2}} \lesssim \| f \|_{H_{prod}^1},$$

where  $d\mu_l$  is a measure as in Proposition 4.1.

Given (5.3) we show the same bound for the nondyadic maximal function by a standard argument. After a straightforward application of Lemma 2.4 we may assume that  $\eta$  has support in  $(-2^{-5}, 2^{-5})$  and vanishes in  $(-2^{-6}, 2^{-6})$ . Let  $\tilde{\eta}$  be supported in  $\cup \pm (2^{-8}, 2^{-3})$  and equal to 1 on  $\cup \pm (-2^{-7}, 2^{-2})$ . We use a Fourier expansion and write for  $1/2 \leq s \leq 2$

$$\frac{1}{s} \eta\left(\frac{t}{s}\right) = \tilde{\eta}(t) \sum_{k \in \mathbb{Z}} c_k(s) e^{2\pi i k t}$$

where  $c_k(s) = O((1 + |k|)^{-N})$  uniformly in  $s \in [1/2, 2]$ . We set

$$d\mu_{k,l} = \int f(t, |t|^m) 2^l \tilde{\eta}(2^l t) e^{2\pi i k 2^l t} dt.$$

and  $M_k f(x) = \sup_l |f * d\mu_{k,l}|$ . An application of (5.3) shows that  $M_k$  maps  $H^1$  to  $L^{1,2}$  with norm  $O((1 + |k|)^4)$  and since  $Mf(x) \lesssim \sum_k (1 + |k|)^{-N} M_k f(x)$  we obtain the inequality for the nondyadic maximal operator from another application of Lemma 2.4.

Now we turn to the proof of (5.3). As in [5] the idea is to approximate  $d\mu_l$  by  $Q_l(Id - \mathcal{P}_l)d\mu_l$  in order to apply Proposition 4.1 and (5.2).

Using straightforward integration by parts arguments we observe that the functions  $\mathcal{P}_0 d\mu_0$  and  $(Id - \mathcal{P}_0)(Id - Q_l)d\mu_0$  are Schwartz functions. By rescaling this, using (1.6), we see that the maximal functions  $\sup_l |f * \mathcal{P}_l d\mu_l|$  and  $\sup_l |f * (Id - \mathcal{P}_l)(Id - Q_l)d\mu_l|$  are dominated by nonisotropic version of the grand maximal function (with respect to (1.6)) which maps  $H_{parabolic}^1$  and hence  $H_{prod}^1$  to  $L^1$ . It thus suffices to show that

$$\| \sup_l |f * (Id - \mathcal{P}_l) Q_l d\mu_l| \|_{L^{1,2}} \lesssim \| f \|_{H_{prod}^1}.$$

Writing  $f_l = (Id - \mathcal{P}_l)Q_l f$ , we can dominate the left-hand side by the  $L^{1,2}$  norm of the square-function  $(\sum_l |f_l * d\mu_l|^2)^{1/2}$ . With this choice of  $f_l$  the inequality

$$(5.4) \quad \left\| \left( \sum_l |d\mu_l * f_l|^2 \right)^{1/2} \right\|_{L^{1,2}} \lesssim \|f\|_{H_{prod}^1}$$

follows from from Proposition 4.1, the embedding  $H_{prod}^1(\ell^2) \subset H_{parabolic}^1(\ell^2)$  and (5.2).

Now consider the analytic family  $H_\gamma$  (and in particular the Hilbert transform  $H = H_0$ ). We may decompose

$$H_\gamma f = \sum_l f * d\sigma_l^\gamma$$

where

$$\langle d\sigma_l^\gamma, f \rangle = \int f(t, |t|^m) 2^l \chi(2^l t) |t|^{\gamma_1 + \gamma_2 m} \frac{dt}{t}$$

and  $\chi(t) = \chi_0(t) - \chi_0(t/2)$ . Note that  $\chi$  is an even function. The functions  $\mathcal{P}_0 d\sigma_0^\gamma$  and  $(Id - \mathcal{P}_0)(Id - Q_l)d\sigma_0^\gamma$  are Schwartz functions as before, but also have mean zero and so their Fourier transforms decay at 0 as well as infinity.

Summing this, we see that  $\mathcal{D}^\gamma \sum_l (Id - \mathcal{P}_l)(Id - Q_l)d\sigma_l$  and  $\mathcal{D}^\gamma \sum_l \mathcal{P}_l d\sigma_l$  are standard product type Calderón-Zygmund kernels and so convolution with these kernels will preserve  $L_p$ ,  $1 < p \leq 2$  and  $H_{prod}^1$ . It thus suffices to show that

$$(5.5) \quad \left\| \sum_l (Id - \mathcal{P}_l)Q_l \mathcal{D}^\gamma d\sigma_l^\gamma * f \right\|_{H_{prod}^{1,2}} \lesssim \|f\|_{H_{prod}^1} \quad \text{if } \operatorname{Re}(\gamma_1 + \gamma_2 m) = 0$$

and

$$(5.6) \quad \left\| \sum_l (Id - \mathcal{P}_l)Q_l \mathcal{D}^\gamma d\sigma_l^\gamma * f \right\|_2 \lesssim \|f\|_2 \quad \text{if } \operatorname{Re}(\gamma_1 + \gamma_2 m) = 1/2$$

with constants depending polynomially on  $\gamma$ .

To see (5.6) we note that a standard stationary phase calculation yields that  $|\widehat{d\sigma_0^\gamma}(\xi)| \lesssim (1 + |\xi|)^{-1/2}$ . By scale invariance we obtain the uniform  $L^2$  boundedness of the operators with convolution kernels  $(Id - \mathcal{P}_l)\mathcal{D}^\gamma d\sigma_l^\gamma$  if  $\operatorname{Re}(\gamma_1 + m\gamma_2) = 1/2$ . The inequality (5.6) follows now from the almost orthogonality of the operators  $Q_l$ .

In order to prove (5.5) it suffices to show that

$$(5.7) \quad \left\| \left( \sum_{k_1, k_2} \left| \sum_l (Id - \mathcal{P}_l)Q_l L_{k_1, k_2} f * d\sigma_l^\gamma \right|^2 \right)^{1/2} \right\|_{L^{1,2}} \lesssim \left\| \left( \sum_{k_1, k_2} |L_{k_1, k_2} f|^2 \right)^{1/2} \right\|_1,$$

by the square function characterization of  $H_{prod}^{1,2}$ ; here  $L_{k_1, k_2}$  are as in §2. For each  $k_1, k_2$  there are at most  $O(1)$  indices  $l$  for which  $(Id - \mathcal{P}_l)Q_l L_{k_1, k_2}$  does not vanish, so we may majorize the left-hand side of (5.7) by

$$\left\| \left( \sum_{k_1, k_2} \sum_l |(Id - \mathcal{P}_l)Q_l L_{k_1, k_2} f * d\sigma_l^\gamma|^2 \right)^{1/2} \right\|_{L^{1,2}}.$$

By Proposition 4.1 we may majorize this in turn by

$$\left\| \{(Id - \mathcal{P}_l)Q_l L_{k_1, k_2} f\}_{l, k_1, k_2 \in \mathbb{Z}} \right\|_{H_{parabolic}^1(\ell^2)}.$$

But this is bounded by  $\|f\|_{H^1_{prod}}$ , by standard arguments similar to the proof of (5.2) above. This concludes the proof of Theorem 1.5. To see that the Hilbert transform  $H$  maps  $H^1_{prod}$  to  $L^{1,2}$  we use in addition the product version of inequality (2.7).

Finally we prove Corollary 1.6. Define the measures  $d\nu_l^\alpha$  by

$$\int f d\nu_l^\alpha = \int f(t, |t|^m) 2^l (\chi(2^l t)) \eta(t) |t|^{m\alpha} \frac{dt}{t}$$

and set  $d\nu_l = d\nu_l^{1/m}$ . We use duality and prove that convolution with  $(Id - \Delta)^{1/2m} \sum_l d\nu_l$  maps  $L^{m'}$  to  $L^{m',2}$ .

It is easy to see that for  $\theta_1 + \theta_2 < 1$ ,  $\theta_1 \geq 0$ ,  $\theta_2 \geq 0$  the functions  $(Id - \Delta)^{\theta/2} \sum_l (Id - \mathcal{P}_l)(Id - Q_l) d\nu_l * f$  and  $(Id - \Delta)^{\theta/2} \sum_l \mathcal{P}_l d\nu_l * f$  are dominated by a constant times the nonisotropic Hardy-Littlewood maximal function of  $f$ .

Let  $\tilde{Q}_l = \tilde{q}_l(D)$  is defined similarly as  $Q_l$  but with  $q_l \tilde{q}_l = q_l$ . Observe that in view of the compact support of  $\eta$  we have  $d\nu_l^\alpha = 0$  if  $l > C_1$  for suitable  $C_1$ . Moreover, if  $l \leq C_1$ , we see, using the definition of  $Q_l$  and the Marcinkiewicz multiplier theorem that for  $\alpha \geq 0$ , that

$$\|(Id - \Delta)^{\alpha/2} (Id - \mathcal{P}_l) \tilde{Q}_l g\|_{L^{m',2}} \lesssim \|\mathcal{D}_2^\alpha Q_l g\|_{L^{m',2}}.$$

Thus it remains to show that

$$\|\{\mathcal{D}_2^\alpha Q_l d\nu_l^\alpha * f\}\|_{H_{prod}^{p,2}(\ell^2)} \lesssim \|f\|_{H_{prod}^p}, \quad \text{Re}(\alpha) = 1 - 1/p,$$

for  $1 \leq p \leq 2$ . This is done by a reprise of the arguments above.

## 6. Rough homogeneous kernels: Preliminary reductions

Let  $\chi_0$  be a radial bump function which is 1 on  $\{x : |x| \leq 1/2\}$  and zero on  $\{x : |x| > 1\}$ , and  $\chi(x) = \chi_0(x) - \chi_0(x/2)$  is then a function on the unit annulus. We also denote by  $\tilde{\chi}(t)$  the restriction of  $\chi$  to the positive real line  $\mathbb{R}^+$ .

In what follows we shall work with the Littlewood-Paley operators introduced in Lemma 2.1 (with  $r = 3$ ) and decompose the identity as  $Id = \sum_k L_0^k L_1^k L_2^k L_3^k$ ; we assume that the numbers  $N_0, \varepsilon$  in Lemma 2.1 are chosen so that  $N_0 \geq 100d$  and  $\varepsilon \leq 10^{-10d}$ .

Let  $\delta_j$  be the dilation operator defined by

$$\delta_j g(x) = 2^{-jd} g(2^{-j}x),$$

and let  $\mathcal{A}$  be the averaging operator defined by

$$\mathcal{A}g(x) = C^{-1} \int \tilde{\chi}(t) t^{-d} g(t^{-1}x) \frac{dt}{t},$$

where  $C = \int \tilde{\chi}(t) \frac{dt}{t}$  is a normalization constant.

Since  $K$  is homogeneous of degree  $-d$  we have the decomposition

$$(6.1) \quad K = \sum_j \delta_j \mathcal{A}[K\chi].$$

If the restriction  $\Omega$  of  $K$  to the unit sphere belongs to  $L \log^2 L(S^{d-1})$  then  $K\chi \in L \log^2 L(\mathbb{R}^d)$  and, since standard Calderón-Zygmund operators map  $L \log^2 L$  to  $L \log L$  the  $L \log^2 L$  assumption for  $K\chi$  is implied by

$$(6.2) \quad \left( \sum_k |L_0^k(K\chi)|^2 \right)^{1/2} \in L \log L.$$

In the present and subsequent section we prove the following stronger version of Theorem 1.1.

**Theorem 6.1.** Let  $K$  be homogeneous of degree  $-d$  and assume that the restriction  $\Omega$  to  $S^{d-1}$  is an integrable function satisfying  $\int \Omega d\sigma = 0$ . Suppose that (6.2) holds. Then the operator  $T_\Omega$  maps  $H^1$  boundedly to  $L^{1,2}$  and also to the Hardy-Lorentz space  $H^{1,2}$ .

We also have

**Theorem 6.2.** Let  $K_0(r\theta) = \tilde{\chi}(r)\Omega(\theta)$  and assume  $\Omega \in L^1(S^{d-1})$  and  $(\sum_k |L_0^k(K_0)|^2)^{1/2} \in L \log L$ . Then  $M_\Omega$  maps  $H^1$  boundedly to  $L^{1,2}$ .

We shall prove Theorem 6.1. To prove Theorem 6.2 we use the argument in §5 to reduce to a version which involves only dyadic dilations. The proof of the relevant estimate for this dyadic maximal operator is similar to the proof of Theorem 6.1 and therefore omitted.

Let  $\mathcal{T}$  be the operator defined by

$$(6.3) \quad \mathcal{T}f = \sum_j \delta_j \mathcal{A}[K\chi] * f$$

We now have to show that  $\mathcal{T}$  is bounded from  $H^1$  to  $H^{1,2}$ . The  $H^1 \rightarrow L^{1,2}$  boundedness follows then from (2.7) and limiting arguments. In our proof of (6.3) we shall assume that the sum in  $j$  is actually finite, but prove a bound which is independent of this finiteness assumption. Again a limiting argument proves the general case.

We now decompose  $f$  in the standard manner as  $f = \sum c_I a_I$ , where  $c_I$  are nonnegative constants such that  $\sum_I c_I \lesssim \|f\|_{H^1}$ , and  $a_I$  is an atom supported on  $I$  with mean zero and such that  $\|a_I\|_\infty \lesssim |I|^{-1}$  ([26]). The center of the atom will be denoted by  $x_I$  and we may assume that each atom has sidelength  $2^{i_I}$  where  $i_I$  is an integer.

For technical reasons we wish to suppress low frequencies in our atoms. Let

$$\tilde{a}_I = \sum_{l \geq -C_0} L_0^{l-i_I} L_1^{l-i_I} L_2^{l-i_I} L_3^{l-i_I} a_I,$$

We assume

$$\left\| \left( \sum_k |L_0^k(K\chi)|^2 \right)^{1/2} \right\|_{L \log L} \leq 1$$

(working with the norm  $\|g\|_{L \log^\gamma L} = \inf\{\lambda > 0 : \int \frac{|g(t)|}{\lambda} \log^\gamma(e + \frac{|g(x)|}{\lambda}) dx \leq 1\}$ ) and we shall prove that

$$(6.4) \quad \left\| \sum_I c_I \sum_j \delta_j \mathcal{A}[(K\chi)] * \tilde{a}_I \right\|_{H^{1,2}} \leq B \sum_I c_I$$

where  $B$  is a constant depending only on  $d$ . Now the cancellation of the atoms shows that  $\|a_I - \tilde{a}_I\|_{H^1} \lesssim 2^{-\varepsilon C_0}$ , and so

$$(6.5) \quad \left\| f - \sum_I c_I \tilde{a}_I \right\| \lesssim 2^{-\varepsilon C_0} \|f\|_{H^1}.$$

Let  $\|\mathcal{T}\|$  denote the  $H^1 \rightarrow H^{1,2}$  operator-norm, which because of our finiteness assumptions is a priori finite. (6.5) implies

$$\|\mathcal{T}f\|_{H^{1,2}} \lesssim 2^{-\varepsilon C_0} \|\mathcal{T}\| \|f\|_{H^1} + B \sum_I c_I.$$

Therefore, if  $C_0$  in the definition of the  $\tilde{a}_I$  is chosen large enough, this implies that  $\|\mathcal{T}\| \lesssim B$ .

In what follows we may assume

$$(6.6) \quad \sum_I c_I \leq 1.$$

We now dispose of the contributions when  $j \leq i_I + 2C_0$ . We claim this portion is not only in  $H^{1,2}$  but is actually in  $H^1$ . Since  $H^1$  is a Banach space we may restrict ourselves to a single cube  $I$ , so that it suffices to show that

$$\left\| \sum_{j \leq i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * \tilde{a}_I \right\|_{H^1} \lesssim 1.$$

This we rewrite as

$$\left\| \sum_{l \geq -C_0} L_0^{l-i_I} L_1^{l-i_I} L_2^{l-i_I} \left[ \sum_{j \leq i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * L_3^{l-i_I} a_I \right] \right\|_{H^1} \lesssim 1.$$

By the analogue of (2.6) for the Littlewood-Paley operators  $L_0^k L_1^k L_2^k$  it thus suffices to show

$$\left\| \left( \sum_{l \geq -C_0} \left| \sum_{j \leq i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * L_3^{l-i_I} a_I \right|^2 \right)^{1/2} \right\|_1 \lesssim 1.$$

Since the expression inside the norm is supported in a fixed dilate of  $I$ , it suffices by the Cauchy-Schwarz inequality to bound

$$\left\| \left( \sum_{l \geq -C_0} \left| \sum_{j \leq i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * L_3^{l-i_I} a_I \right|^2 \right)^{1/2} \right\|_2 \lesssim |I|^{-1/2}.$$

By modifying the method of rotations argument in [4] we see that the operator with convolution kernel  $\sum_{j < i_I + 2C_0} \delta_{j \leq i_I + s} [K\chi]$  is bounded on  $L^2$ ; hence the above reduces to

$$(6.7) \quad \left( \sum_{l \geq -C_0} \|L_3^{l-i_I} a_I\|_2^2 \right)^{1/2} \lesssim |I|^{-1/2}.$$

But this follows from the  $L^2$  estimates on  $a_I$  and the almost orthogonality of the  $L_3^{l-i_I}$ .

We now turn to the contributions  $j > i_I + 2C_0$  and we wish to establish

$$\left\| \sum_I c_I \sum_{l \geq -C_0} L_0^{l-i_I} L_1^{l-i_I} L_2^{l-i_I} \left[ \sum_{j > i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * L_3^{l-i_I} a_I \right] \right\|_{H^{1,2}} \lesssim 1.$$

We set  $a_{I,l} = L_3^{l-i_I} a_I$  and let  $\{e_j\}$  be the standard orthonormal basis of unit vectors in  $\ell^2$ . By the remark following Lemma 2.1 we reduce to show that

$$\left\| \sum_I c_I \sum_{l \geq -C_0} L_1^{l-i_I} \left[ \sum_{j > i_I + 2C_0} \delta_j \mathcal{A}[K\chi] * L_2^{l-i_I} a_{I,l} \right] e_{l-i_I} \right\|_{L^{1,2}(\ell^2)} \lesssim 1.$$

By Lemma 2.1 we may decompose

$$K\chi = S_1^0(K\chi) + \sum_{k=1}^{\infty} L_1^k L_0^k(K\chi).$$

One easily checks that the convolution operator with kernel  $K = \sum_j \delta_j \mathcal{A}[S_1^0 K\chi]$  is a standard Calderón-Zygmund operator. Indeed using the cancellation of the functions  $L_2^{l-i_I} a_{I,l}$  it is easy to see that for a fixed cube  $I$

$$\left\| \left( \sum_{l \geq -C_0} \left| \sum_{j > i_I + 2C_0} \delta_j [\mathcal{A}S_1^0(K\chi)] * L_2^{l-i_I} a_{I,l} \right|^2 \right)^{1/2} \right\|_1 \lesssim 1,$$

and the resulting  $H^1 \rightarrow L^1(\ell^2)$  inequality follows for this part.

Therefore it suffices to prove that

$$(6.8) \quad \left\| \sum_I c_I \sum_{j>i_I+2C_0} \sum_{l\geq-C_0} L_1^{l-i_I} \delta_j \mathcal{A}(\sum_{k>0} L_1^k K^k) * L_2^{l-i_I} a_{I,l} e_{l-i_I} \right\|_{L^{1,2}(\ell^2)} \lesssim 1,$$

where still  $a_{I,l} = L_3^{l-i_I} a_I$ , and  $K^k = L_0^k(K\chi)$ .

We can rewrite the desired estimate for this portion using the identity

$$L_1^m \delta_j = \delta_j L_1^{j+m}.$$

Consequently we have to prove for  $q = 2$  the inequality

$$(6.9) \quad \begin{aligned} & \left\| \sum_I \sum_{j>2C_0+i_I} \sum_{l\geq-C_0} c_I \delta_j (L_1^{l-i_I+j} \mathcal{A}[\sum_{k>0} L_1^k K^k]) * L_2^{l-i_I} a_{I,l} e_{l-i_I} \right\|_{L^{1,q}(\ell^q)} \\ & \lesssim \sup_I |I|^{1-1/q} \left( \sum_l \|a_{I,l}\|_q^q \right)^{1/q} \left\| \left( \sum_k |K^k|^q \right)^{1/q} \right\|_{L \log^{2-\frac{2}{q}} L} \end{aligned}$$

for arbitrary measurable functions  $K^k$  on  $\{x : 1/4 \leq |x| \leq 4\}$  and  $a_{I,l}$  on  $CI$ . (6.8) follows then by using also (6.7).

We shall deduce the inequality for  $q = 2$  from the inequality (6.9) for  $q = 1$  and the obvious modification of (6.9) for  $q = \infty$ .

Notice that

$$(6.10) \quad \begin{aligned} \|L_1^{l-i_I+j} \mathcal{A}[L_1^k K^k]\|_{L^1 \rightarrow L^1} & \leq \int |\tilde{\chi}(t)| t^{-d} \|\psi_1^{l-i_I+j} * t^{-d} \psi_1^k(t^{-1}\cdot)\|_1 \|K^k\|_1 dt \\ & \lesssim 2^{-|l-i_I+j-k|} \|K_k\|_1 \end{aligned}$$

where we have used the cancellation of the Littlewood-Paley kernels. The last estimate immediately implies (6.9) for  $q = 1$ . The nontrivial part concerns the estimate for  $q = \infty$  which is proved in the next section. From these two estimates we deduce (6.9) for  $q = 2$  by complex interpolation, using Lemma 2.2. Assuming

$$\left\| \left( \sum_k |K^k|^2 \right)^{1/2} \right\|_{L \log L} \leq 1,$$

we consider the analytic family  $K_z = \{K_z^k\}_{k \in \mathbb{Z}}$  defined by

$$K_z^k(x) = K^k(x) |K^k(x)|^{1-2z} |K(x)|_{\ell^2}^{2z-1} [\log(e + |K(x)|_{\ell^2})]^{1-2z}$$

if  $K^k(x) \neq 0$  and by  $K_z^k(x) = 0$  otherwise. Then  $\|K_{i\tau}\|_{L^1(\ell^1)} \lesssim 1$  and  $\|K_{1+i\tau}\|_{L \log^2 L(\ell^\infty)} \lesssim 1$ . The rest is straightforward.

## 7. Rough homogeneous kernels: The weak type estimate

We are now proving the analogue of (6.9) for  $q = \infty$ . In addition to (6.6) we may also suppose that

$$(7.1) \quad \sup_I \sup_l \|a_{I,l}\|_\infty \leq 1, \quad \left\| \sup_k |K_k| \right\|_{L \log^2 L} \leq 1$$

and show that for  $\alpha > 0$

$$(7.2) \quad \text{meas} \left( \left\{ x : \left| \sum_I \sum_{j>2C_0+i_I} \sum_{l \geq -C_0} c_I \delta_j(L_1^{l-i_I+j} \mathcal{A}[L_1^k K^k]) * L_2^{l-i_I} a_{I,l} e_{l-i_I} \right|_{\ell^\infty} > \alpha \right\} \right) \lesssim \alpha^{-1}.$$

Let  $F = \sum_I c_I \frac{\chi_I}{|I|}$ . Since  $\|F\|_1 \lesssim 1$ , we may apply the standard dyadic Calderón-Zygmund decomposition to  $F$  at level  $\alpha$ , and obtain a collection of disjoint dyadic cubes  $\mathcal{J} = \{J\}$  such that  $\sum_J |J| \lesssim \alpha$ ,  $\int_J F(x) dx \lesssim \alpha |J|$ , and such that  $F$  is  $O(\alpha)$  outside of  $\bigcup_J J$ .

To every dyadic cube  $I$  we assign a nonnegative integer  $t_I$  as follows. If  $I$  is not contained in any of the  $J$ , then  $t_I = 0$ . If  $I$  is a subset of a cube  $J \in \mathcal{J}$ , then  $t_I$  is chosen so that the sidelength of  $J$  is  $2^{t_I}$  times the sidelength of  $I$ . One can view  $t_I$  as a stopping time; roughly speaking,  $2^{t_I} I$  is the largest dilate of  $I$  on which the mean of  $F$  is greater than  $\alpha$ , or  $I$  if no such dilate exists.

The contribution of the terms in (7.2) for which  $j < i_I + t_I + 2C_0$  is contained inside the exceptional set  $\bigcup_J C_J$ , which has measure  $O(\alpha)$ . We can therefore restrict ourselves to the case  $j \geq i_I + t_I + 2C_0$ . We change the summation variable to  $s = j - i_I - t_I \geq 2C_0$ . Thus for the expression

$$(7.3) \quad \mathcal{E}(x) = \sum_I \sum_{s \geq 2C_0} \sum_l c_I \sum_{k>0} \delta_{i_I+t_I+s}(L_1^{l+s+t_I} \mathcal{A}[L_1^k K^k]) * L_2^{l-i_I} a_{I,l}(x) e_{l-i_I}$$

we have to show that the measure of the set  $\{x : |\mathcal{E}(x)|_{\ell^\infty} > \alpha\}$  is  $O(\alpha^{-1})$ . This will be estimated by further splitting the expression  $\mathcal{E}(x)$  into four pieces and then by applying of Chebyshev's inequality and  $L^1$  or  $L^2$  estimates for the individual pieces.

We now describe this splitting. Let

$$(7.4) \quad M(x) = \sup_{k>0} |K^k(x)|.$$

We break up the functions  $K^k$  into a bounded part and an integrable part (this truncation has first been used in [9]). Let  $\varepsilon_0 > 0$  be a constant to be chosen later ( $\varepsilon_0 = 10^{-2}$ , say, works). For all  $k$  write  $K^k = 2^{\varepsilon_0(s+l)} K_{l,s,I}^k + R_{l,s,I}^k$ , where  $|K_{l,s,I}^k(x)| \leq 1$  and the remainder  $R_{l,s,I}^k$  is the restriction of  $K^k$  to the set  $\{x : M(x) \geq 2^{\varepsilon_0(s+l)}\}$ . We split

$$\mathcal{E}(x) = \mathcal{E}_1(x) + \mathcal{E}_2(x) + \mathcal{E}_3(x) + \mathcal{E}_4(x)$$

where

(7.5.1)

$$\mathcal{E}_1(x) = \sum_I \sum_{s \geq 2C_0} \sum_{l \geq -C_0} c_I \sum_{\substack{k>0 \\ |k-l-s-t_I| \geq s+l}} \delta_{i_I+t_I+s}(L_1^{l+s+t_I} \mathcal{A}[L_1^k K^k]) * L_2^{l-i_I} a_{I,l}(x) e_{l-i_I}$$

(7.5.2)

$$\mathcal{E}_2(x) = \sum_I \sum_{s \geq 2C_0} \sum_{l \geq -C_0} c_I \sum_{\substack{k>0 \\ |k-l-s-t_I| < s+l}} \delta_{i_I+t_I+s}(L_1^{l+s+t_I} \mathcal{A}[L_1^k R_{l,s,I}^k]) * L_2^{l-i_I} a_{I,l}(x) e_{l-i_I}$$

(7.5.3)

$$\mathcal{E}_3(x) = \sum_I \sum_{l \geq 2C_0} \sum_{2C_0 \leq s \leq l} c_I 2^{\varepsilon_0(s+l)} \sum_{\substack{k>0 \\ |k-l-s-t_I| < s+l}} \delta_{i_I+t_I+s}(L_1^{l+s+t_I} \mathcal{A}[L_1^k K_{l,s,I}^k]) * L_2^{l-i_I} a_{I,l}(x) e_{l-i_I}$$

(7.5.4)

$$\mathcal{E}_4(x) = \sum_I \sum_{s \geq 2C_0} \sum_{-C_0 \leq l < s} c_I 2^{\varepsilon_0(s+l)} \sum_{\substack{k>0 \\ |k-l-s-t_I| < s+l}} \delta_{i_I+t_I+s}(L_1^{l+s+t_I} \mathcal{A}[L_1^k K_{l,s,I}^k]) * L_2^{l-i_I} a_{I,l}(x) e_{l-i_I}$$

It suffices to show that for  $i = 1, 2, 3, 4$  the measure of the set  $\{x : |\mathcal{E}_i(x)|_{\ell^\infty} > \alpha/4\}$  is  $O(\alpha^{-1})$ . By Chebyshev's inequality and the continuous imbedding  $\ell^1 \subset \ell^2 \subset \ell^\infty$  it suffices to show that

$$(7.6) \quad \|\mathcal{E}_1\|_{L^1(\ell^1)} + \|\mathcal{E}_2\|_{L^1(\ell^1)} + \|\mathcal{E}_3\|_{L^1(\ell^1)} \lesssim 1$$

and

$$(7.7) \quad \|\mathcal{E}_4\|_{L^2(\ell^2)} \lesssim \alpha.$$

The estimation of  $\mathcal{E}_1$  and  $\mathcal{E}_2$  is straightforward. Since  $\|(L_1^{l+s+t_I} \mathcal{A}[L_1^k K^k])\|_{L^1 \rightarrow L^1} \lesssim 2^{-|k-l-s-t_I|}$  we get

$$(7.8) \quad \begin{aligned} \|\mathcal{E}_1\|_{L^1(\ell^1)} &\lesssim \sum_I \sum_{s \geq 2C_0} \sum_{l \geq -C_0} c_I \sum_{|k-l-s-t_I| \geq s+l} 2^{-|k-l-s-t_I|} \|L_2^{l-i_I} a_{I,l}\|_1 \\ &\lesssim \sum_I c_I \sum_{s \geq 2C_0} \sum_{l \geq -C_0} 2^{-s-l} \lesssim 1. \end{aligned}$$

Next, by the definition of  $R_{l,s,I}^k$

$$\|L_1^{l+s+t_I} \mathcal{A}[L_1^k R_{l,s,I}^k]\|_1 \lesssim 2^{-|k-l-s-t_I|} \int_{x: M(x) \geq 2^{\varepsilon_0(s+l)}} M(x) dx$$

and therefore

$$(7.9) \quad \begin{aligned} \|\mathcal{E}_2\|_{L^1(\ell^1)} &\lesssim \sum_I \sum_{s \geq 2C_0} \sum_{l \geq -C_0} c_I \sum_{|k-l-s-t_I| \leq s+l} 2^{-|k-l-s-t_I|} \int_{x: M(x) \geq 2^{\varepsilon_0(s+l)}} M(x) dx \\ &\lesssim \sum_I c_I \int |M(x)| \log^2(e + |M(x)|) dx \lesssim 1. \end{aligned}$$

The following Lemma is crucial for the estimation of  $\mathcal{E}_3$ .

**Lemma 7.1.** *Suppose that  $g$  is a bounded function supported in  $\{x : 1/4 \leq |x| \leq 4\}$  and  $a$  is supported in a cube  $I$  with sidelength  $2^{i_I}$ ; moreover  $\|a\|_\infty \leq |I|^{-1}$ . Then for  $m \geq 0$*

$$\|\delta_{i_I+m}[L_1^{l+m} \mathcal{A}g] * a\|_1 \lesssim 2^{-l/2} \|g\|_\infty$$

**Proof.** We may assume  $\|g\|_\infty \leq 1$ . Let  $\mathcal{V}_m = \{\nu\}$  be a maximal  $2^{-m}$ -separated subset of unit vectors in  $\mathbb{R}^d$ ; its cardinality is  $O(2^{m(d-1)})$ . We may split  $g = \sum_\nu g_{m,\nu}$  where  $g_{m,\nu}$  is supported in the sector  $\{x : |\frac{x}{|x|} - \nu| \lesssim 2^{-m+10}\}$  (and in the annulus where  $1/4 \leq |x| \leq 4$ ).

Now  $\delta_{i_I+m}[L_1^{l+m} \mathcal{A}g] * a$  is supported in a rectangle of dimensions  $C_1 2^{i_I} \times \dots \times C_1 2^{i_I} \times C_1 2^{i_I+m}$ . Therefore by the Cauchy-Schwarz inequality

$$(7.10) \quad \begin{aligned} \|\delta_{i_I+m}[L_1^{l+m} \mathcal{A}g] * a\|_1 &\lesssim \sum_{\nu \in \mathcal{V}_m} 2^{(i_I d + m)/2} \|\delta_{i_I+m}[L_1^{l+m} \mathcal{A}g_{m,\nu}] * a\|_1 \\ &\lesssim |I|^{1/2} 2^{md/2} \left( \sum_{\nu \in \mathcal{V}_m} \|\delta_{i_I+m}[L_1^{l+m} \mathcal{A}g_{m,\nu}] * a\|_2^2 \right)^{1/2}. \end{aligned}$$

We estimate this sum using Plancherel's theorem. For  $\xi \in (\mathbb{R}^d)^*$

$$\begin{aligned} |\widehat{\mathcal{A}g}_{m,\nu}(-\xi)| &= \left| \int_{r=1/4}^4 \int_{\theta} g_{m,\nu}(r\theta) r^{d-1} \int \chi(\tau) e^{i\tau \langle r\theta, \xi \rangle} d\tau d\theta dr \right| \\ &\lesssim \|g\|_{\infty} \int_{1/4}^4 \int_{|\theta-\nu| \leq 2^{-m+10}} (1 + |\langle \theta, \xi \rangle|)^{-N} d\theta dr. \\ &\lesssim 2^{-m(d-1)/2} \left( \int_{|\theta-\nu| \leq 2^{-m+10}} (1 + |\langle \theta, \xi \rangle|)^{-2N} d\theta \right)^{1/2}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{\nu \in \mathcal{V}_m} \|\delta_{I+m}[L_1^{l+m} \mathcal{A}g_{m,\nu}] * a\|_2^2 &\lesssim 2^{-m(d-1)} \sum_{\nu \in \mathcal{V}_m} \int |\widehat{\psi_1^{l+m}}(2^{i_I+m}\xi)|^2 \int_{|\theta-\nu| \leq 2^{-m+10}} (1 + |\langle \theta, 2^{i_I+m}\xi \rangle|)^{-2N} d\theta |\widehat{a}(\xi)|^2 d\xi \\ &\lesssim 2^{-m(d-1)} \int |\widehat{\psi_1^{l+m}}(2^{i_I+m}\xi)|^2 \int_{S^{d-1}} |(1 + |\langle \theta, 2^{i_I+m}\xi \rangle|)^{-2N} d\theta |\widehat{a}(\xi)|^2 d\xi \\ &\lesssim 2^{-m(d-1)} \int |\widehat{\psi_1}(\frac{\xi}{2^{l-i_I}})|^2 \min\{1, 2^{-i_I-m}|\xi|^{-1}\} |\widehat{a}(\xi)|^2 d\xi \\ (7.11) \quad &\lesssim 2^{-m(d-1)} 2^{-(m+l)} \|\widehat{a}\|_2^2 \lesssim 2^{-md-l} |I|^{-1}, \end{aligned}$$

by Plancherel's theorem and the estimate  $|\widehat{\psi_1}(\xi)| \lesssim \min\{|\xi|^2, |\xi|^{-2}\}$ .

The asserted estimate follows from (7.10) and (7.11).  $\square$

We now estimate the  $L^1(\ell^1)$  norm of  $\mathcal{E}_3$ . To apply Lemma 7.1 we note that  $L_2^{l-i_I} a_{I,l}$  is supported in a fixed dilate of  $I$  and  $\|L_2^{l-i_I} a_{I,l}\|_{\infty} \lesssim |I|^{-1}$ . Moreover  $\|L_1^k K_{l,s,I}^k\|_{\infty} \lesssim 1$ , uniformly in  $k, l, s, I$ . Hence

$$(7.12) \quad \|\mathcal{E}_3\|_{L^1(\ell_1)} \lesssim \sum_I c_I \sum_{l \geq 2C_0} \sum_{2C_0 \geq s \leq l} 2^{\varepsilon_0(s+l)} \sum_{\substack{k>0 \\ |k-l-s-t_I| < s+l}} 2^{-l/2} \|L_1^k K_{l,s,I}^k\|_{\infty} \lesssim 1.$$

Finally we turn to the estimation of  $\|\mathcal{E}_4\|_{L^2(\ell^2)}$ . We first observe the basic estimate

**Lemma 7.2.**

$$\left\| \sum_I c_I \frac{\chi_{2^{t_I} I}}{|2^{t_I} I|} \right\|_2 \lesssim \alpha^{1/2}.$$

**Proof.** Consider first those cubes  $I$  for which  $t_I = 0$ . It is easy to see that this contribution is bounded pointwise by  $\min(F, C\alpha)$  for some constant  $C$ , and so the claim follows since  $\|F\|_1 \lesssim 1$ .

Now consider the cubes  $I$  for which  $t_I > 0$ . This part is majorized pointwise by

$$\left\| \sum_J \chi_{CJ} \right\|_2 \lesssim \left\| \sum_J \chi_J \right\|_2 = \left( \sum_J |J| \right)^{1/2} \lesssim \alpha^{1/2},$$

where for the first inequality we have used Lemma 2.3.  $\square$

The claimed estimate for  $\mathcal{E}_4$  will follow from

**Lemma 7.3.** Let  $g_I$  be bounded and supported on  $\{x : 1/4 \leq |x| \leq 4\}$  and set  $b_{I,l} = L_2^{l-i_I} a_{I,l}$ . Assume  $l \geq -C_0$ ,  $s \geq 0$ . Then for suitable  $\varepsilon > 0$

$$\left\| \sum_I c_I \delta_{i_I+t_I+s} (L_1^{l+s+t_I} \mathcal{A} g_I) * b_{I,l} \right\|_2 \lesssim \sup_I \|g_I\|_\infty 2^{-s\varepsilon} \alpha^{1/2}.$$

**Proof.** This inequality is closely related to one in [25] and we shall adapt the proof here. Let  $\mathcal{V}_s = \{\nu\}$  be a maximal  $2^{-s}$ -separated subset of the unit sphere  $S^{d-1}$ ; the cardinality of this set is  $O(2^{(d-1)s})$ . We decompose  $g_I = \sum_\nu g_{I,\nu}$ , where each  $g_{I,\nu}$  is a bounded function on the sector

$$(7.13) \quad \mathfrak{S}_\nu^s = \{x : 1/4 \leq |x| \leq 4, \angle(x, \nu) \leq 2^{-s}\};$$

here we used  $\angle(x, \nu)$  to denote the angle  $x$  and  $\nu$  make at the origin.

We introduce a localization in Fourier space to a conic neighborhood of the hyperplane perpendicular to  $\nu$ , namely

$$\Sigma_\nu^s = \{\xi : |\langle \xi, \nu \rangle| \leq 2^{-s/2} |\xi|\}$$

(The exact choice of aperture  $2^{-s/2}$  is unimportant as long as it is well between  $2^{-s}$  and 1). We define the multiplier  $Q_\nu^s$  whose symbol  $m_\nu$  is homogeneous of degree 0, and equals 1 on  $\Sigma_\nu^s$  and vanishes outside a slight widening of  $\Sigma_\nu^s$ .

We then reduce to showing that

$$(7.14) \quad \left\| \sum_I c_I \sum_{\nu \in \mathcal{V}_s} Q_\nu^s \delta_{i_I+t_I+s} (L_1^{l+s+t_I} \mathcal{A} g_{I,\nu}) * b_{I,l} \right\|_2 \lesssim \sup_I \|g_I\|_\infty 2^{-s\varepsilon} \alpha^{1/2}$$

and, for fixed  $\nu$ ,

$$(7.15) \quad \left\| \sum_I c_I (Id - Q_\nu^s) \delta_{i_I+t_I+s} (L_1^{l+s+t_I} \mathcal{A} g_{I,\nu}) * b_{I,l} \right\|_2 \lesssim \sup_{I,\nu} \|g_{I,\nu}\|_\infty 2^{-sN} \alpha^{1/2}$$

where  $N \leq N_0/10$  (recall that  $N_0 \geq 100d$ ). The estimate (7.15) is favorable if  $N > d - 1$ .

To prove (7.15) we show the estimate

$$(7.16) \quad |(Id - Q_\nu^s) \delta_{i_I+t_I+s} (L_1^{l+s+t_I} \mathcal{A} g_{I,\nu})(x)| \lesssim \|g_{I,\nu}\|_\infty 2^{-sN} \frac{2^{-(i_I+t_I)d}}{(1 + 2^{-(i_I+t_I)}|x|)^N}$$

for all  $\nu \in \mathcal{V}_s$ . From (7.16) we may estimate

$$|(Id - Q_\nu^s) \delta_{i_I+t_I+s} (L_1^{l+t_I+s} \mathcal{A} g_{I,\nu}) * b_{I,l}| \lesssim 2^{-Ns} H_I * \frac{\chi_{2^{t_I} I}}{|2^{t_I} I|}$$

where  $H_I$  is the  $L^1$  dilate of a radially decreasing  $L^1$  function. By Lemma 2.3 and Lemma 7.2 the left hand side of (7.15) is dominated by

$$2^{-sN} \left\| \sum_I c_I \frac{\chi_{2^{t_I} I}}{|2^{t_I} I|} \right\|_2 \lesssim 2^{-sN} \alpha^{1/2}.$$

We now show (7.16). Fix  $\nu$ . Rescaling so that  $i_I + t_I + s = 0$ , it suffices to show that

$$|L_1^j (Id - Q_\nu^s) \mathcal{A} h(x)| \lesssim 2^{-(N+d)s} \|h\|_{L^\infty(\mathfrak{S}_\nu^s)} (1 + |x|)^{-N}$$

for all  $j \geq l + t_I + s \geq s$  and all bounded  $h$  supported on  $\mathfrak{S}_\nu^s$ .

Fix  $j, x$ . We expand the left-hand side as

$$\left| (2\pi)^{-d} \int_{\mathfrak{S}_\nu^s} h(z) \iiint (1 - m_\nu(\xi)) e^{i\langle \xi, x - 2^{-j}y - tz \rangle} \psi_1(y) \tilde{\chi}(t) d\xi dy \frac{dt}{t} dz \right|$$

where the moments of  $\psi_1$  vanish up to order  $N_0$  and  $\tilde{\chi}$  is supported where  $1/4 \leq t \leq 4$ . The decay in  $x$  follows from the fact that the phase is non-stationary in the  $\xi$  variable when  $|x| \gg 1$ .

Now we demonstrate the  $2^{-Ns}$  bound; we may assume that  $|x| \ll 2^{s/5}$ . Since  $h$  is supported in  $\mathfrak{S}_\nu^s$  and  $m_\nu$  equals 1 on  $\Sigma_\nu$  we see that for each  $|\xi| \gtrsim 2^j$ , the phase is non-stationary in the  $t$  variable (with a gradient of at least  $2^{\varepsilon s}$ ). For  $|\xi| \lesssim 2^j$  one picks up a loss of  $(2^j/|\xi|)^C$ , but this can be compensated for by the moment conditions on  $\psi_1$ , since  $j \geq s$ .

To show (7.14) we use the fact that the  $Q_\nu^s$  have some weak orthogonality. More precisely, we have for any functions  $f_\nu$  that

$$(7.17) \quad \left\| \sum_\nu Q_\nu^s f_\nu \right\|_2^2 \lesssim 2^{-\varepsilon s} 2^{(d-1)s} \sum_\nu \|f_\nu\|_2^2;$$

as in [25] this estimate is easily proven from Plancherel's theorem, the Cauchy-Schwarz inequality, and geometrical considerations. Because of this orthogonality, and Lemma 7.2, it now suffices to show that

$$(7.18) \quad \left\| \sum_I c_I \delta_{i_I + t_I + s} \mathcal{A} g_{I,\nu} * a_I \right\|_2 \lesssim 2^{-(d-1)s} \left\| \sum_I c_I \frac{\chi_{2^{t_I} I}}{|2^{t_I} I|} \right\|_2,$$

uniformly in  $\nu \in \mathcal{V}_s$ .

Fix  $\nu$ . Let  $R_\nu^s$  be the rectangle centered at the origin, with dimensions  $C_1 2^{-s} \times \dots \times C_1 2^{-s} \times C_1$  so that the long side is parallel to  $\nu$ . Then, if  $C_1$  is chosen large enough there is the uniform pointwise estimate

$$|\delta_{i+t_I+s} [\mathcal{A} g_{I,\nu}] * a_I| \lesssim 2^{-s(d-1)} \|g_{I,\nu}\|_\infty \delta_{i+t_I+s} \left( \frac{\chi_{R_\nu^s}}{|R_\nu^s|} \right) * \frac{\chi_{2^{t_I} I}}{|2^{t_I} I|}.$$

Thus (7.18) follows from Lemma 2.3. This completes the proof of (7.14) and the Lemma.  $\square$

The estimate (7.7) is an immediate consequence of Lemma 7.3. The estimate (7.6) holds by (7.8), (7.9) and (7.12) and thus we have proved the asserted weak type inequality.

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